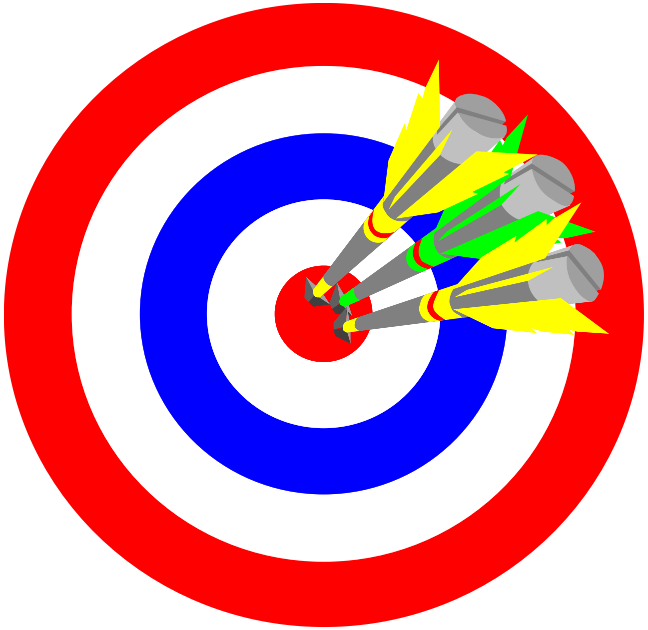
Chapter 12

Areas & Volumes of Solids

Objectives



A. Use the terms defined in the chapter correctly.
B. Properly use and interpret the symbols for the terms and concepts in this chapter.
C. Appropriately apply the theorems in this chapter.
D. Identify the parts of prisms, pyramids, cylinders, cones, spheres.
E. Calculate the total area and volume of prisms, pyramids, cylinders, cones and spheres.
F. Use proportions to compare the perimeters, areas and volumes of solids.

Section 12-1

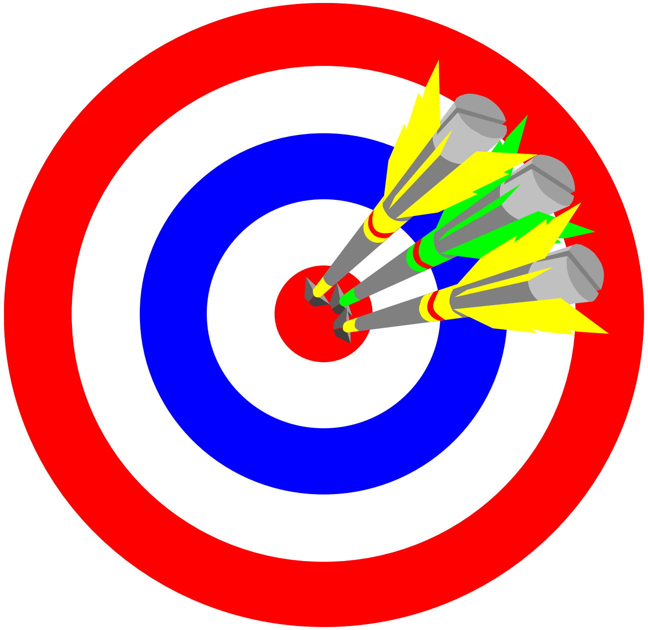
Prisms

Homework Pages 478-480:

2-32 evens

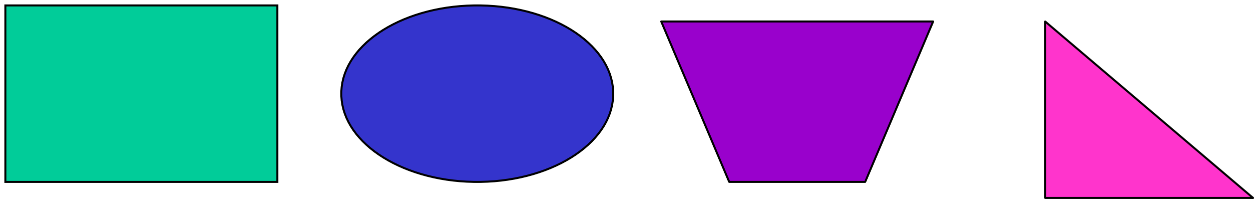
Excluding 6, 12, 16, 20, 26

Objectives



A. Define and identify the parts of a prism.
B. Define and identify different types of prisms.
C. Define and calculate the lateral area of prisms.
D. Define and calculate the volume of prisms.
E. Understand and utilize the theorems of lateral area and volume of prisms to solve real world problems.

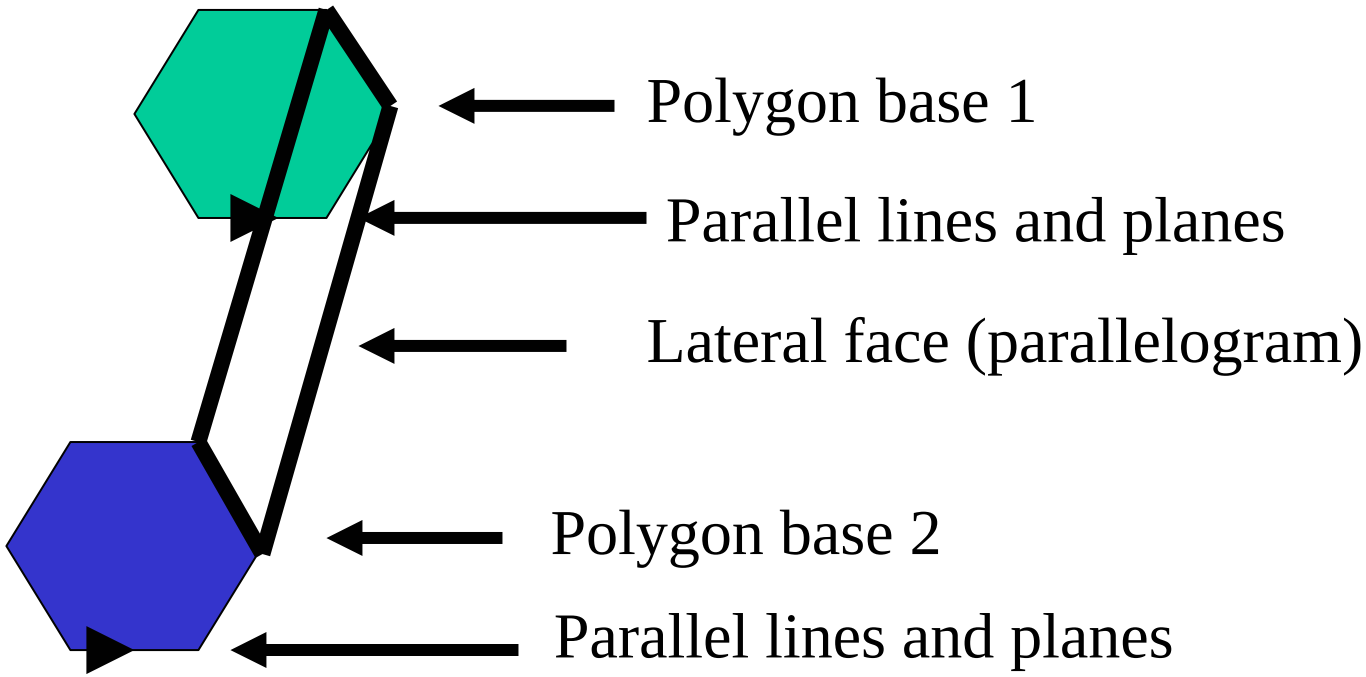
We are leaving the two-dimensional world behind …



We are entering ….

3-dimensional SPACE

- Prism → A geometric solid having two polygon bases and having parallelograms for its lateral faces

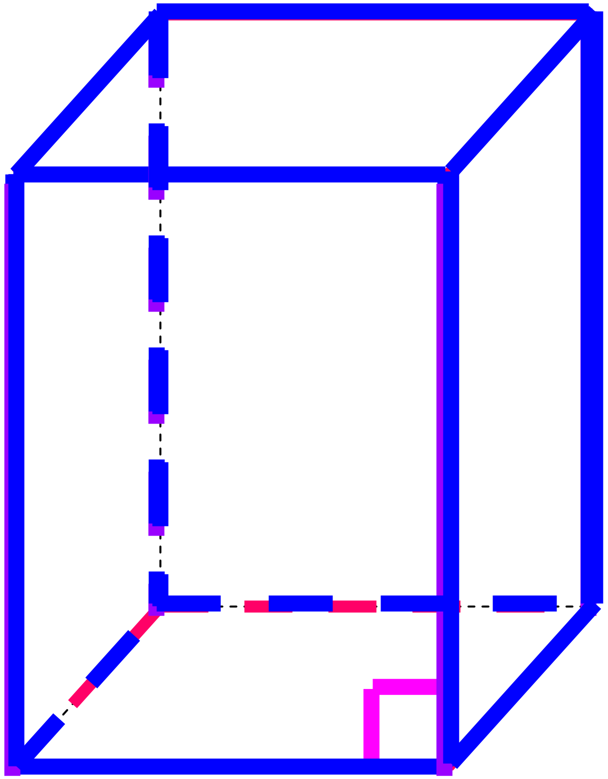


You have been sentenced to …

Life in Prism

- Bases of a prism → Congruent polygons lying in parallel planes
- Altitude of a prism → A segment joining and perpendicular to the base planes
- Lateral faces of a prism → The parallelograms making up the sides of the prism.
- Lateral edges → Intersection of adjacent lateral faces

Parts of a Prism



Bases

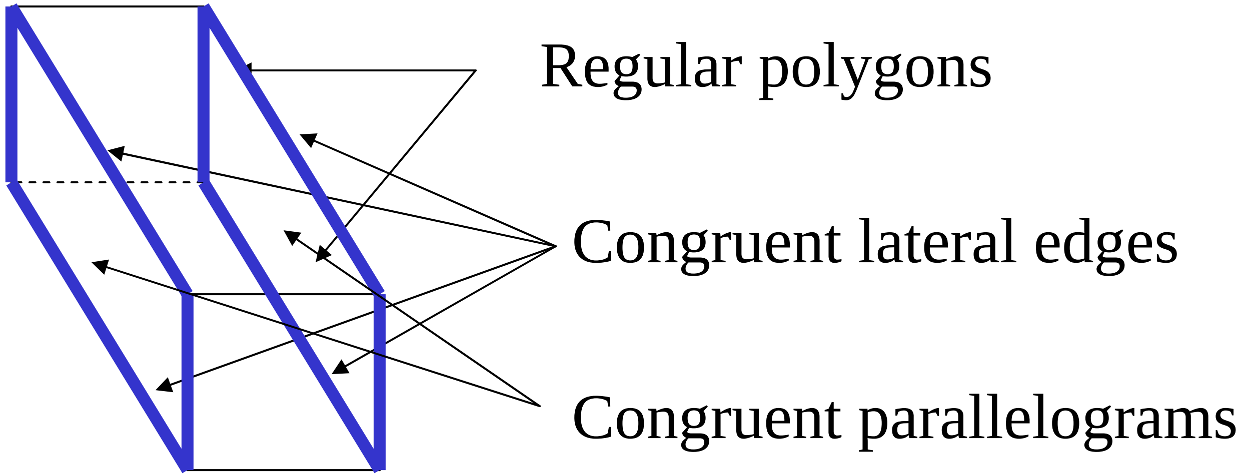
Altitude

Lateral Faces

Lateral Edges

Types of Prisms

- regular prism:
  - bases are regular polygons
  - lateral edges are congruent
  - lateral faces are congruent parallelograms



Types of Prisms

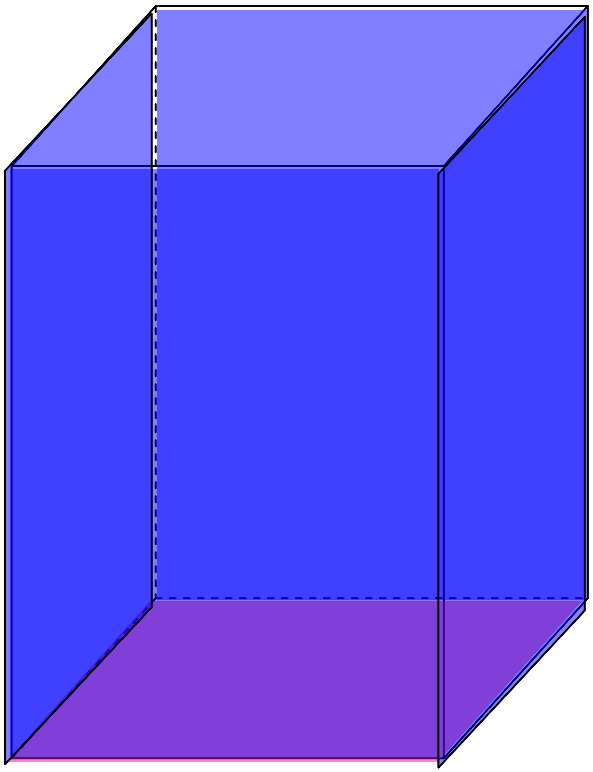
- right prism:
  - lateral faces are rectangles
  - lateral edge is altitude
  - bases must be congruent polygons, but not necessarily regular polygons
- cube: a rectangular solid with square faces
- oblique prism:
  - lateral faces are parallelograms
  - lateral edge is not altitude

Prism Areas

- The concept of the surface area of a 3-dimensional (or solid) object is simple:
  - Imagine you peeled the cover off of the 3-dimensional object.
  - Lay the ‘cover’ flat.
  - Measure the area of the ‘flat cover’ the same way you would measure any 2-dimensional area.
- Lateral area → The sum of the areas of ALL lateral faces.
- Base area → The area of a SINGLE base.

★ Total area of a prism → The sum of the lateral area and the areas of the bases. TA = LA + 2B where LA is the lateral area and B is the area of a base.

Areas of a Prism



Lateral Area

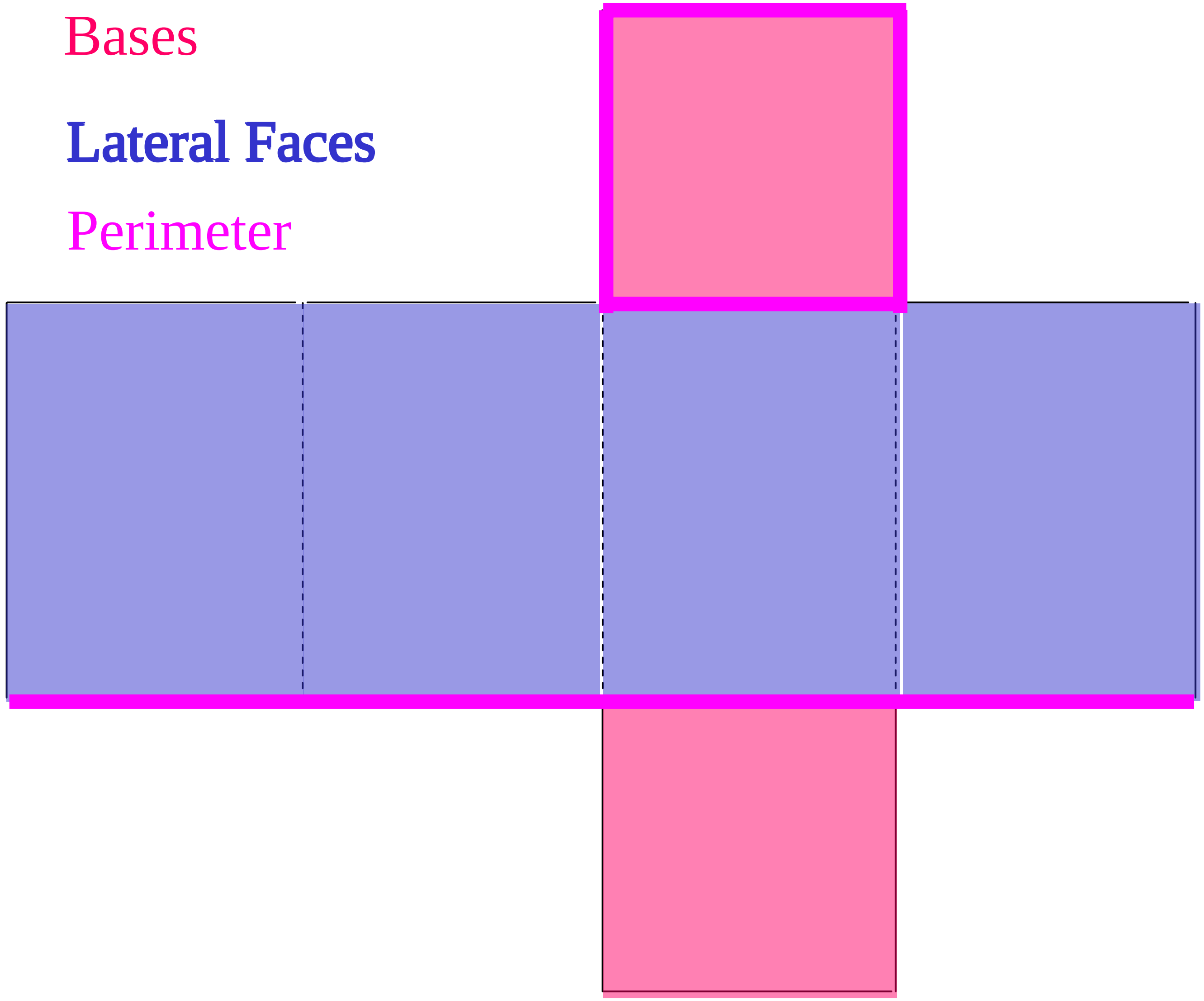
Base Area

Disassembled Square Right Prism

Bases

Lateral Faces

Perimeter

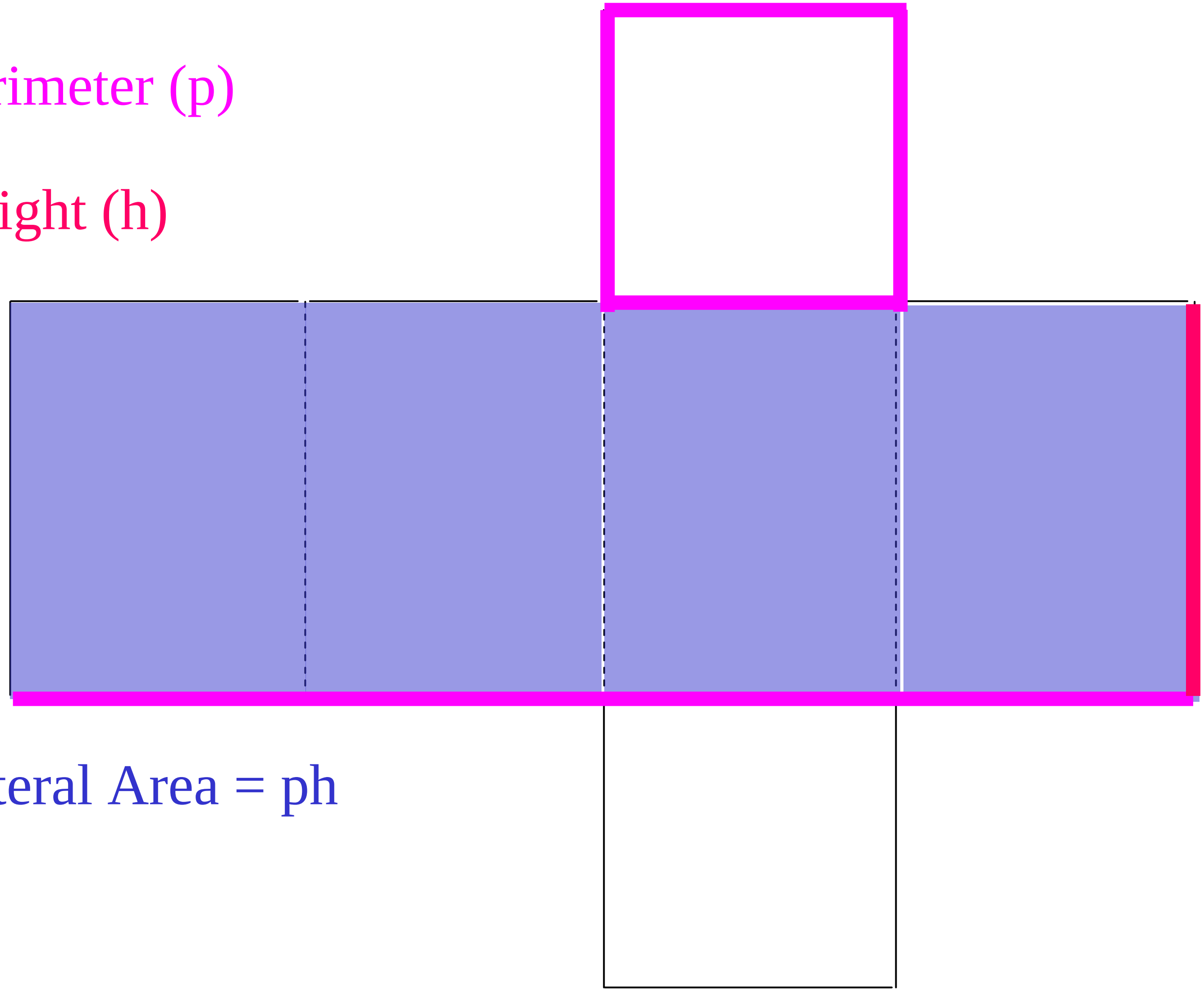


★Theorem 12-1

The lateral area of a right prism equals the perimeter of a base times the height of the prism.

Perimeter (p)

Height (h)



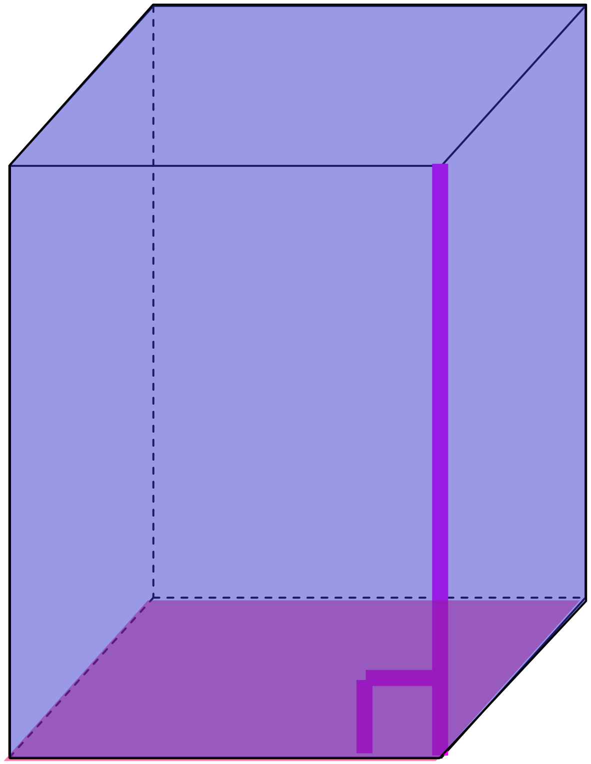
Lateral Area = ph

Volumes of Ideas

- The concept of the volume of a rectangular solid is fairly simple:
  - Find the area of the base (a slice of the solid).
  - Multiply by the number of areas/slices that are stacked together.
- Volume is measure in CUBIC units:
  - $Feet^3$
  - $Centimeters^3$

★ Theorem 12-2

The volume of a right prism equals the area of the base times the height of the prism.



Base Area (B)

Height (h)

Volume = Bh

Sample Problems

rectangular solid	3.
l	6
w	3
h	3
L.A.	$54\,units^2$
T.A.	$90\,units^2$
V	54 units^3

$$V = Bh = length \times width \times height$$

$$54 = 6 \times 3 \times h$$

$$54 = 18h$$

$$h = 3$$

$$L.A. = ph$$

$$p = (2 \times length) + (2 \times width)$$

$$p = (2)(6) + (2)(3) = 18$$

$$L.A. = ph = (18)(3) = 54\,units^2$$

$$T.A. = L.A. + 2 \times B$$

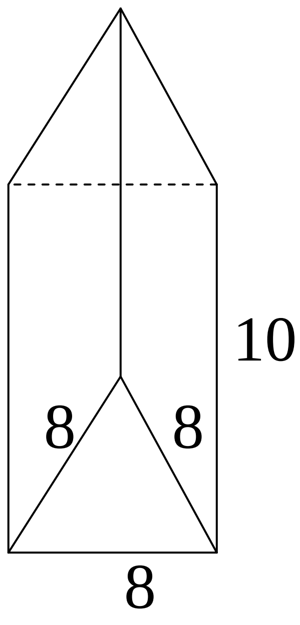
$$B = l \times w = 6 \times 3 = 18$$

$$T.A. = 54 + (2)(18) = 90\,units^2$$

Sample Problems

Draw the figure. Find the lateral area, total area and volume of each right prism with the base described.

17. Equilateral triangle with side 8; h = 10.



Lateral Area = (perimeter base) x height

$$LA = (8 \times 3)(10) = 240\ units^2$$

Total Area = LA + (2 x Base Area)

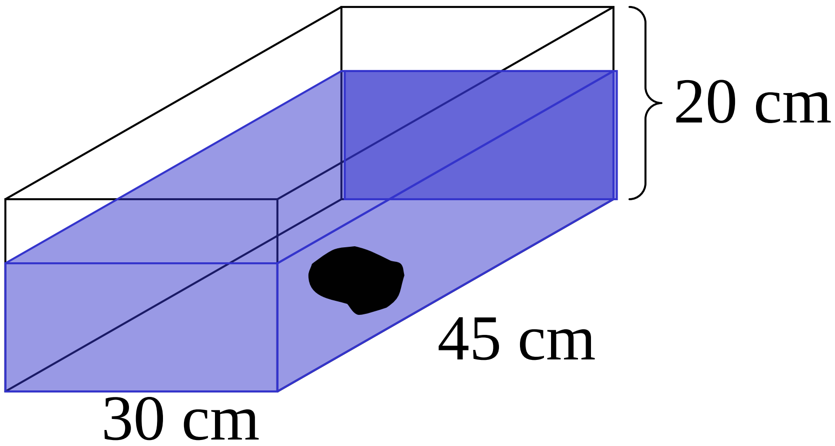
$$TA = 240 + 2 \times \left[\left(\frac{1}{2}\right)(8)\left(4\sqrt{3}\right)\right] = \left(240 + 32\sqrt{3}\right) units^2$$

Volume = Base Area x Height

$$V = \left(16\sqrt{3}\right) \times 10 = 160\sqrt{3}\ units^3$$

Sample Problems

23. The container shown has the shape of a rectangular solid. When a rock is submerged, the water level rises 0.5 cm. Find the volume of the rock.



EASIER than it looks!

The volume of the rock is equal to the volume of the water when it is 0.5 cm high.

$$Volume_{Rock} = Volume_{0.5cm\ water}$$

$$V_{0.5cm\ water} = l \times w \times h = (45)(30)(0.5) = 675\ cm^3$$

Sample Problems

rectangular solid	1.	5.
l	6	9
w	4	
h	2	2
L.A.		60
T.A.		
V		

Sample Problems

cube	7.	9.	11.
e	3		
T.A.			150
V		1000	

13. Find the lateral area of a right pentagonal prism with height 13 and base edges 3.2, 5.8, 6.9, 4.7, and 9.4.

15. If the edge of a cube is doubled, the total area is multiplied by ___ and the volume is multiplied by ____?

Sample Problems

Find the lateral area, total area and volume of each right prism with the base described.

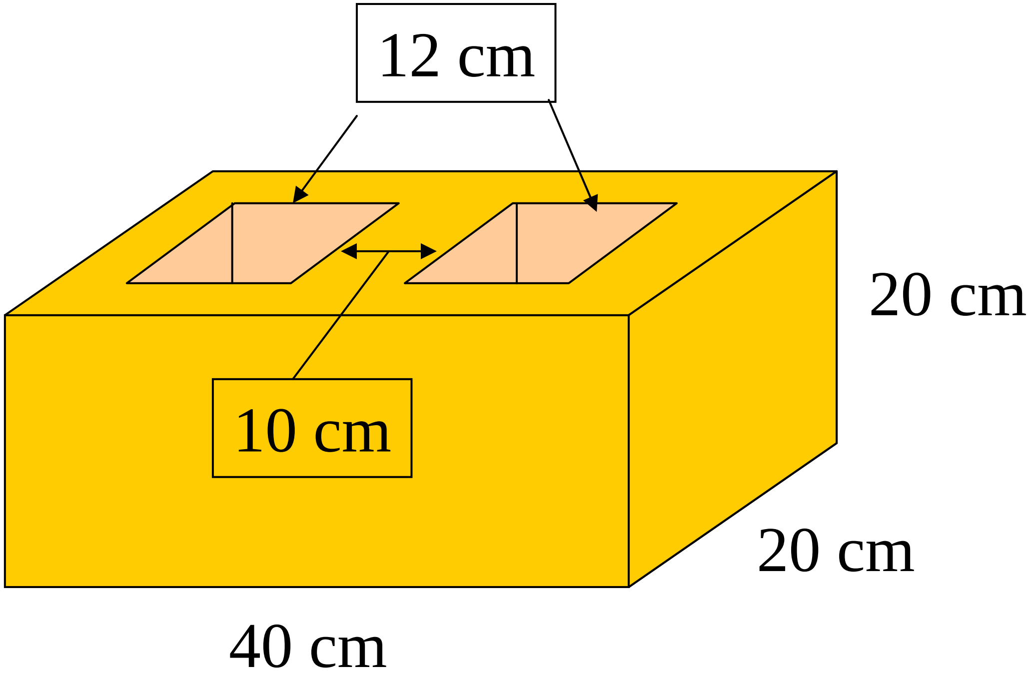
19. Isosceles triangle with sides 13, 13, 10; $h = 7$.

21. Rhombus with diagonals 6 and 8; $h = 9$.

Sample Problems

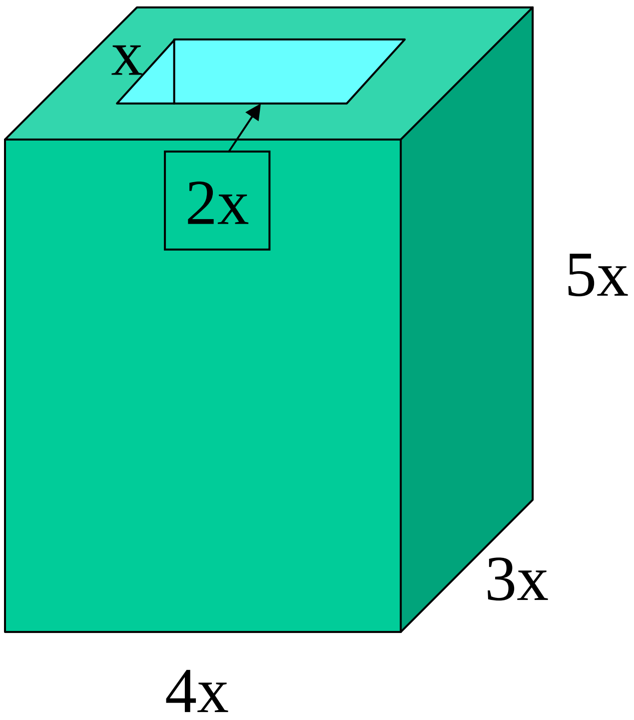
25. A brick with dimensions 20 cm, 10 cm, and 5 cm weighs 1.2 kg. A second brick of the same material has dimensions 25 cm, 15 cm, and 4 cm. What is its weight?

27. Find the weight to the nearest kg, of the cement block shown. Cement weighs 1700 kg/m^3.



Sample Problems

29. Find the volume and the total surface area.



31. The length of a rectangular solid is twice the width, and the height is three times the width. If the volume is 162, find the total area.

Section 12-2

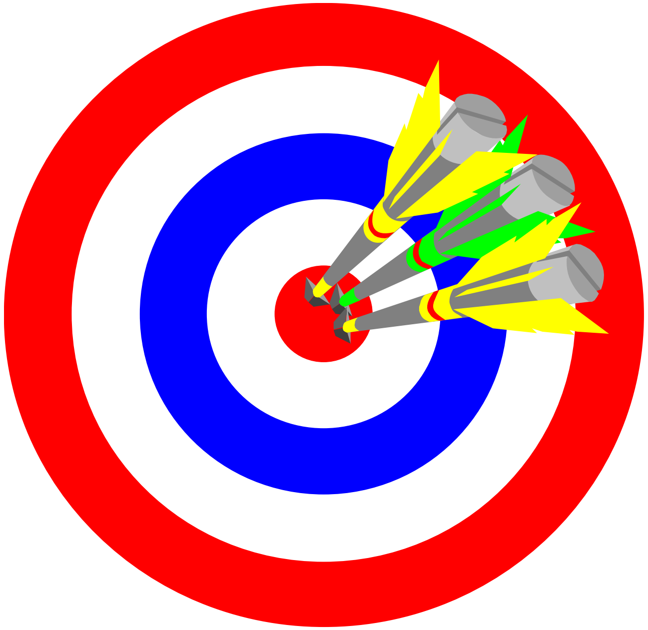
Pyramids

Homework Pages 485-486:

2-26 evens

Excluding 8, 14, 20

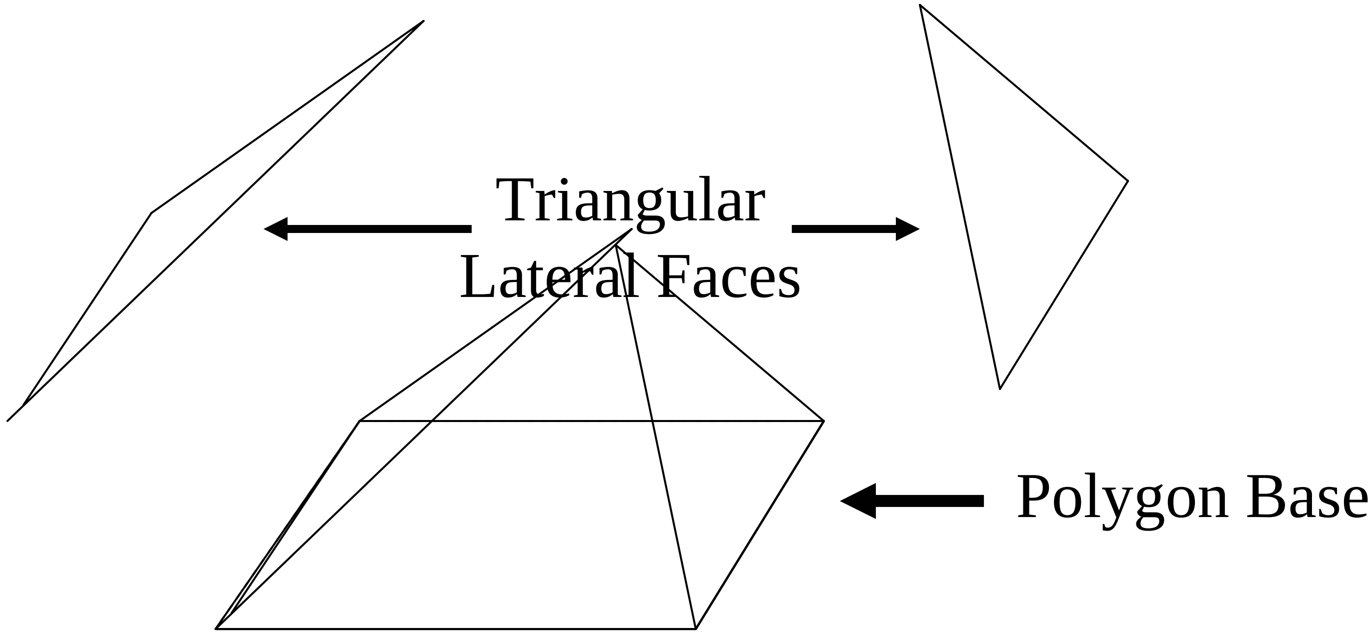
Objectives



A. Define and identify the parts of a pyramid.
B. Define and identify different types of pyramids.
C. Define and calculate the lateral area of pyramids.
D. Define and calculate the volume of pyramids.
E. Understand and utilize the theorems of lateral area and volume of pyramids to solve real world problems.

Pyramids – More than just an Egyptian tomb

- pyramid: a geometric solid having one polygon base and triangular lateral faces.

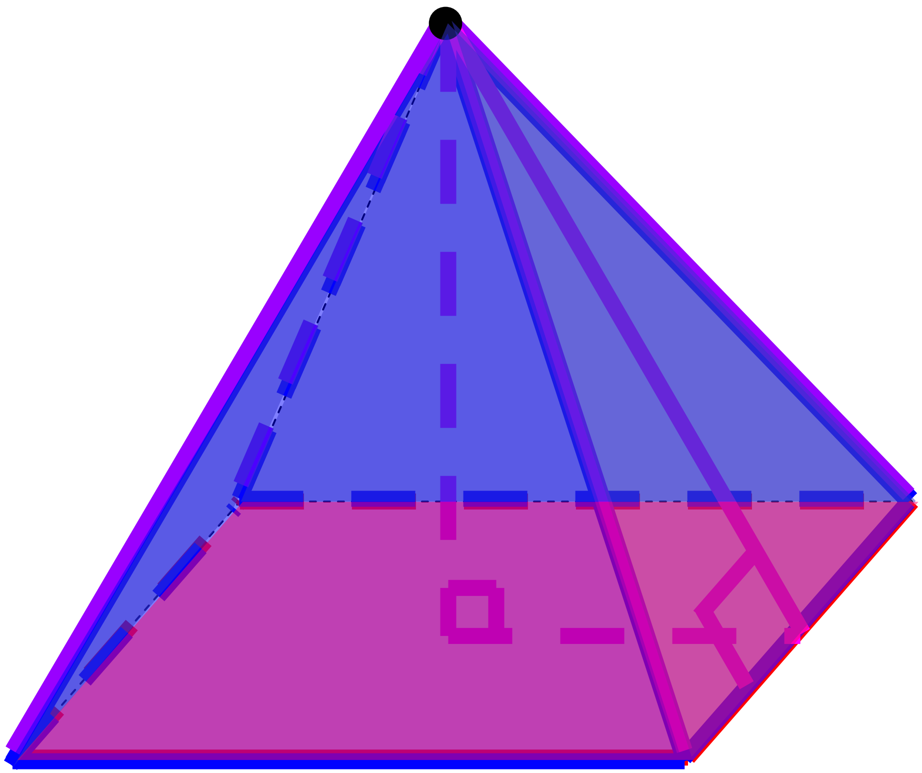


Building Blocks of Pyramids

- vertex: the intersection of all lateral faces
- base of a pyramid: polygon opposite the vertex
- altitude of a pyramid: segment from the vertex perpendicular to the base
- lateral face of a pyramid: triangles making up the sides of the pyramid intersecting at the vertex
- slant height: the height of the lateral face of a regular pyramid

★ total area of a pyramid: the sum of the lateral area and the area of the base. TA = LA + B where LA is the lateral area and B is the area of the base.

Parts of a Pyramid



Vertex

Base

Lateral Faces

Altitude

Lateral Edges

Slant Height

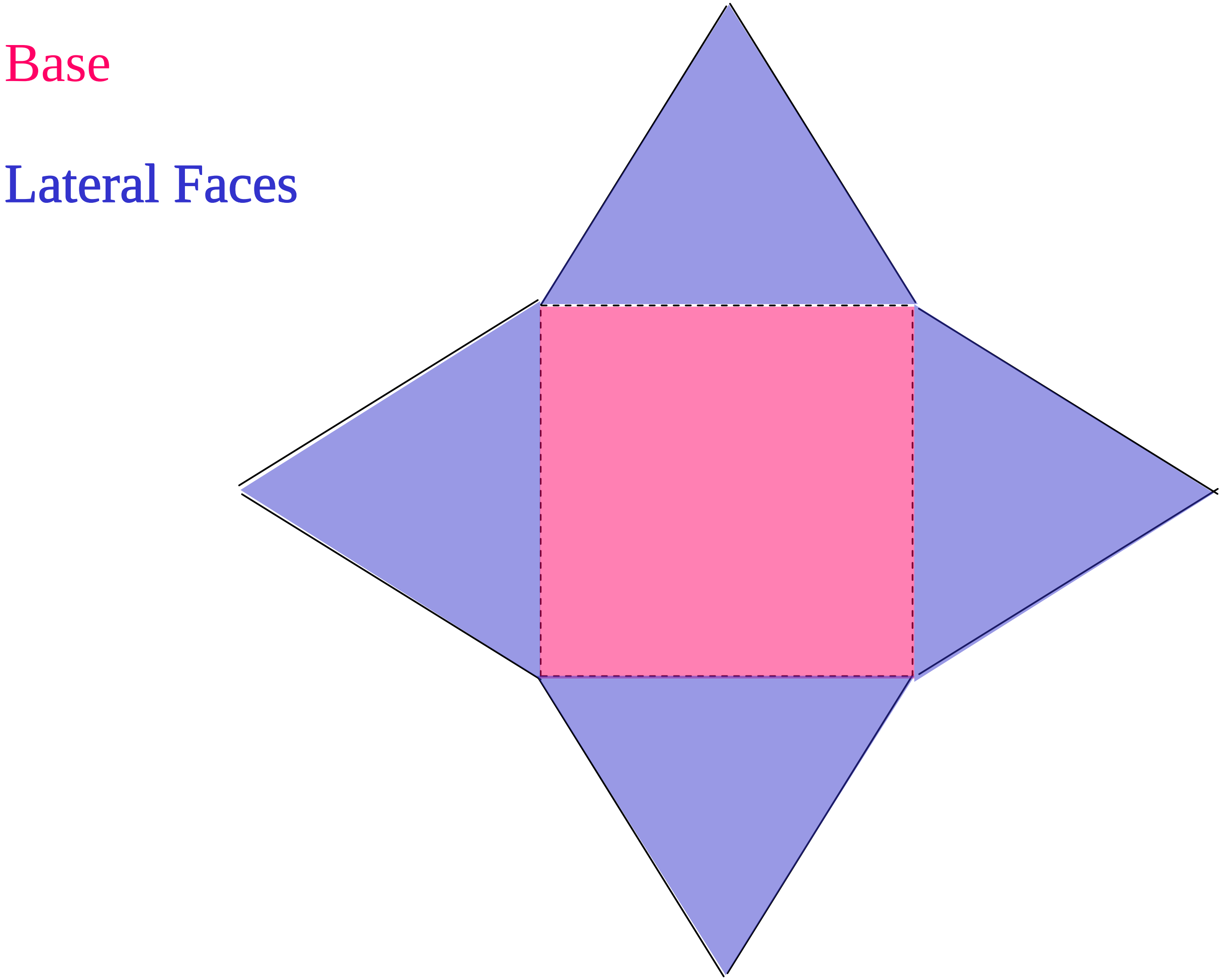
Lateral Area

Base Area

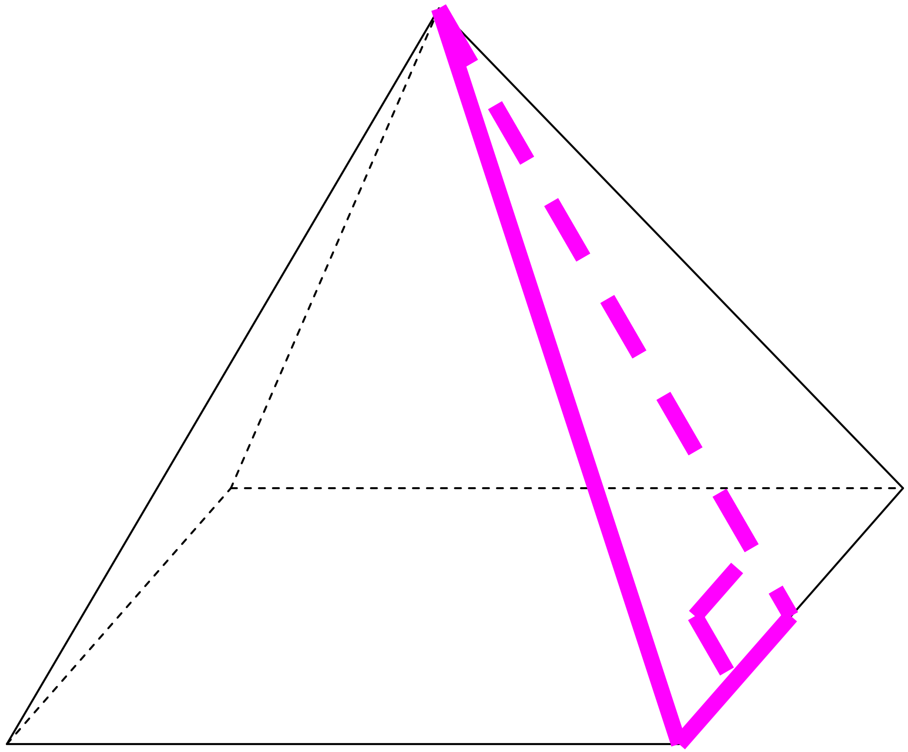
Types of Pyramids

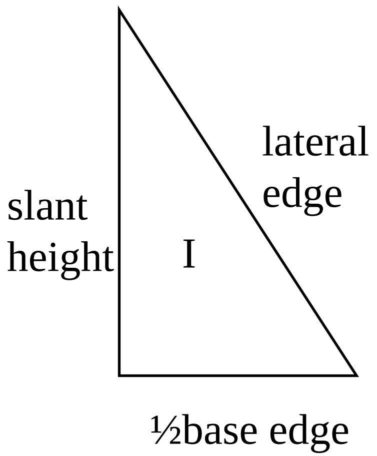
- regular pyramid:
  - base is a regular polygon
  - lateral edges are congruent
  - lateral faces are congruent isosceles triangles
  - altitude intersects the center of the base

Disassembled Square Right Pyramid

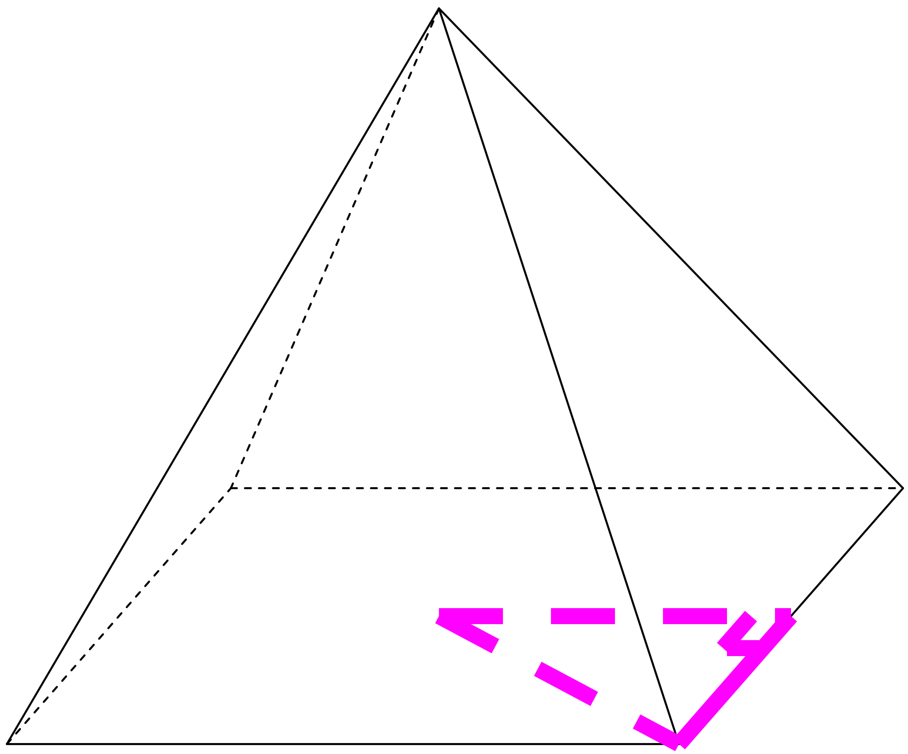


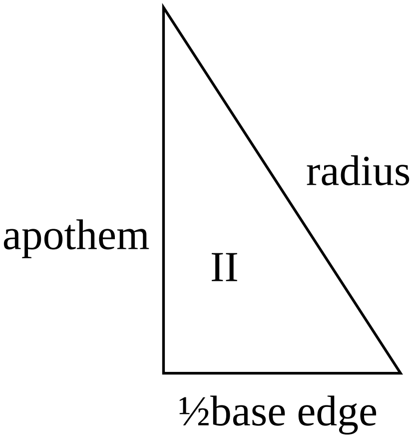
★ Special Right Triangles in a Pyramid



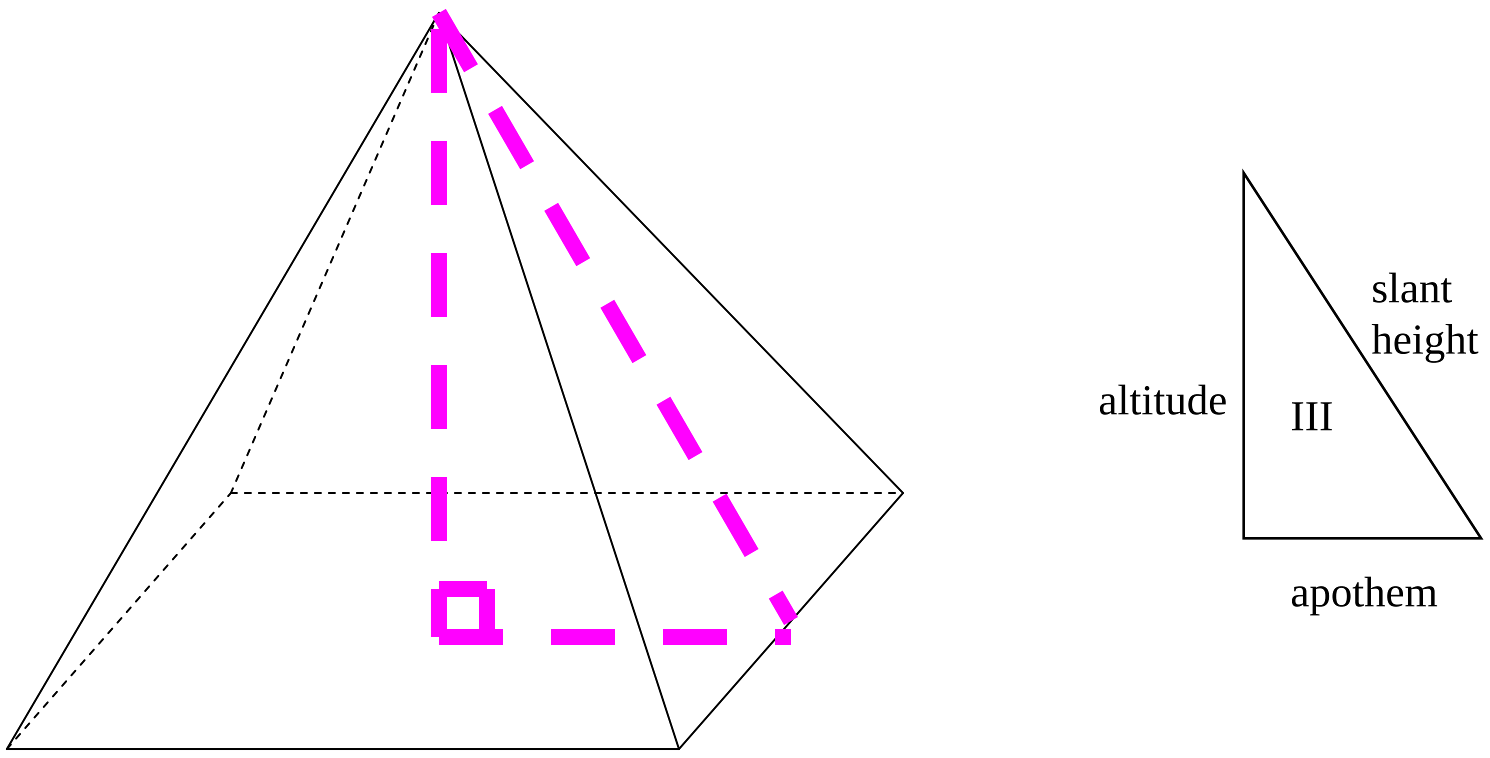


★ Special Right Triangles in a Pyramid

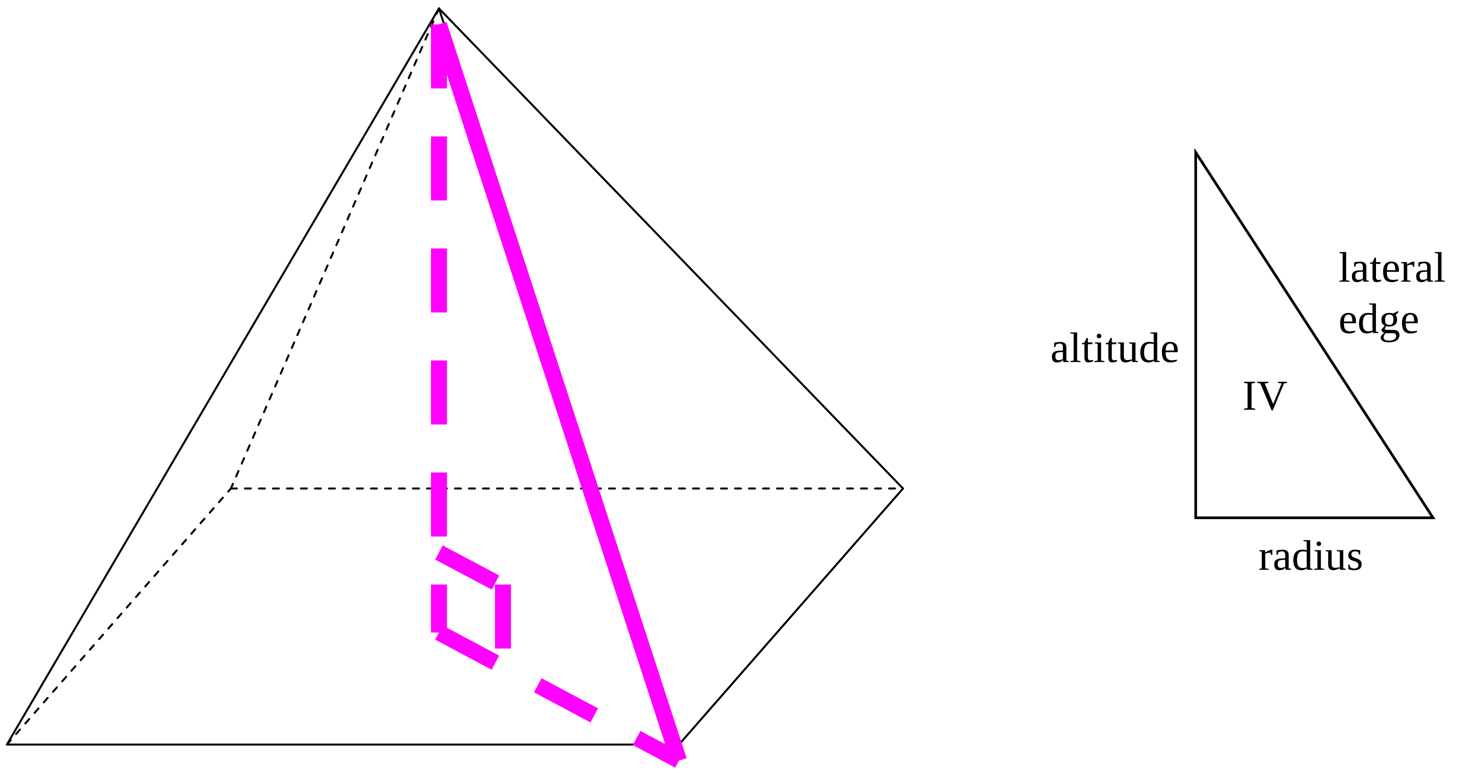




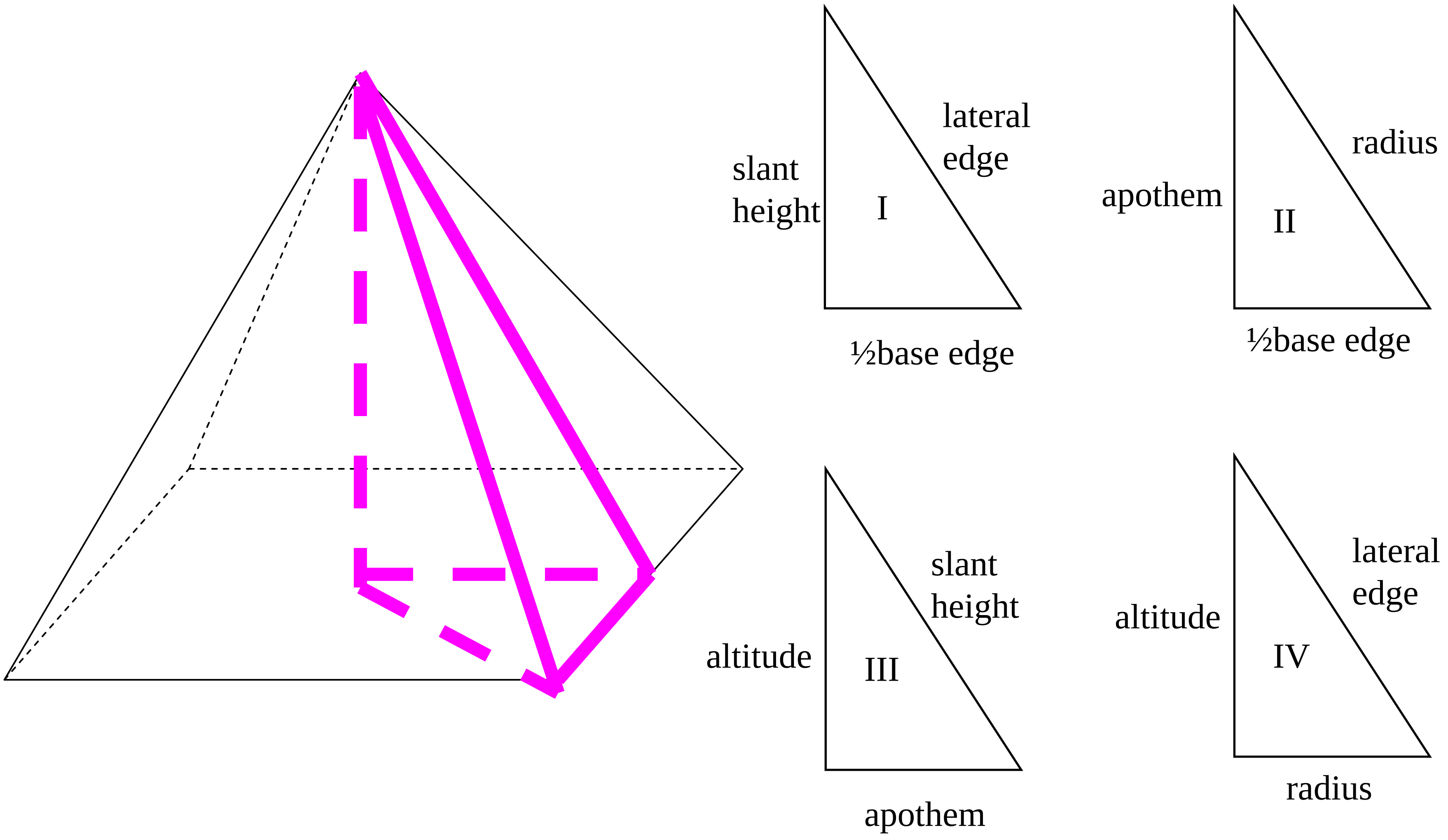
★ Special Right Triangles in a Pyramid



★ Special Right Triangles in a Pyramid



★ Special Right Triangles in a Pyramid

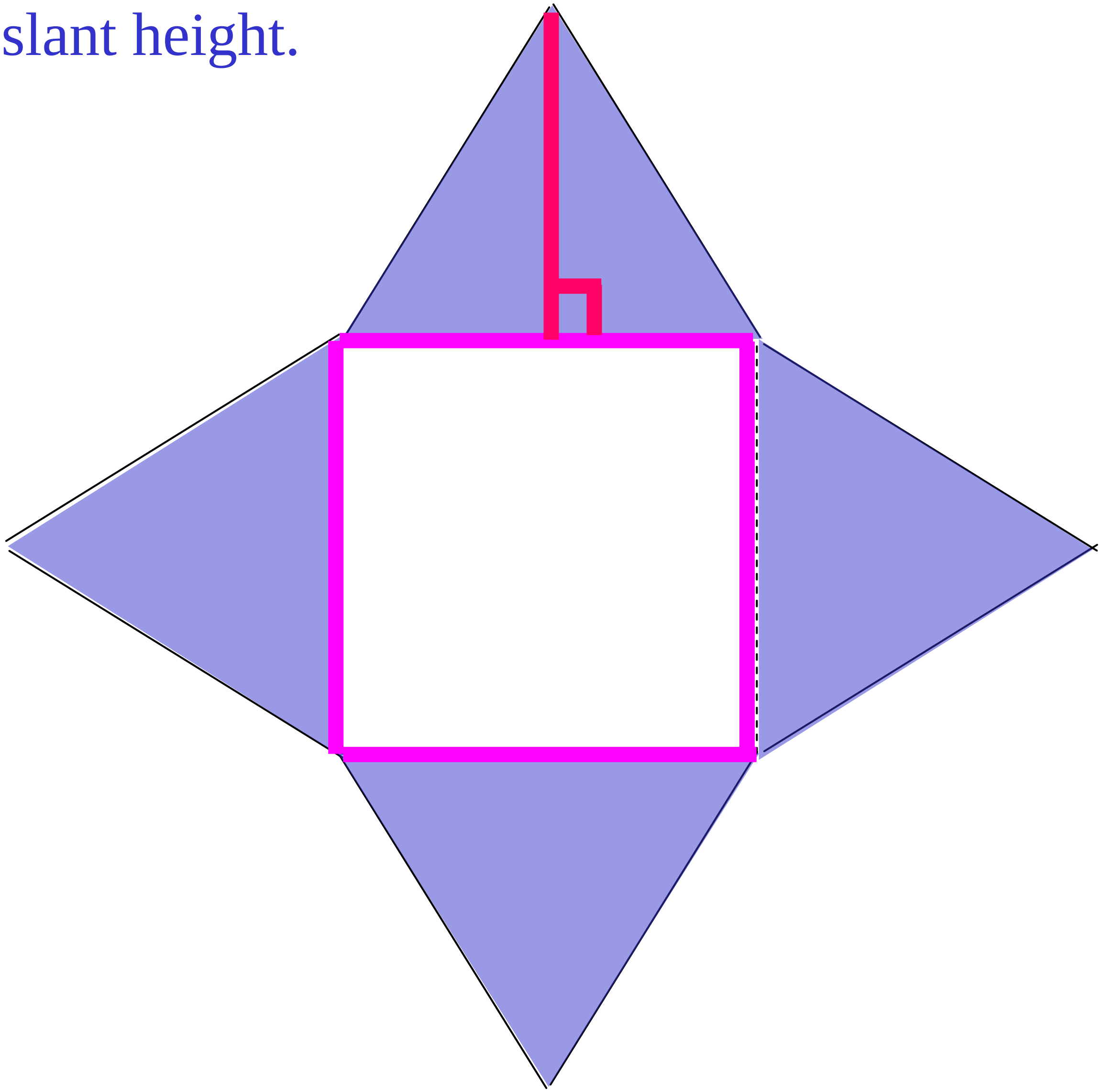


★ Theorem 12-3

The lateral area of a regular pyramid equals half the perimeter of the base times the slant height.

Perimeter (p)

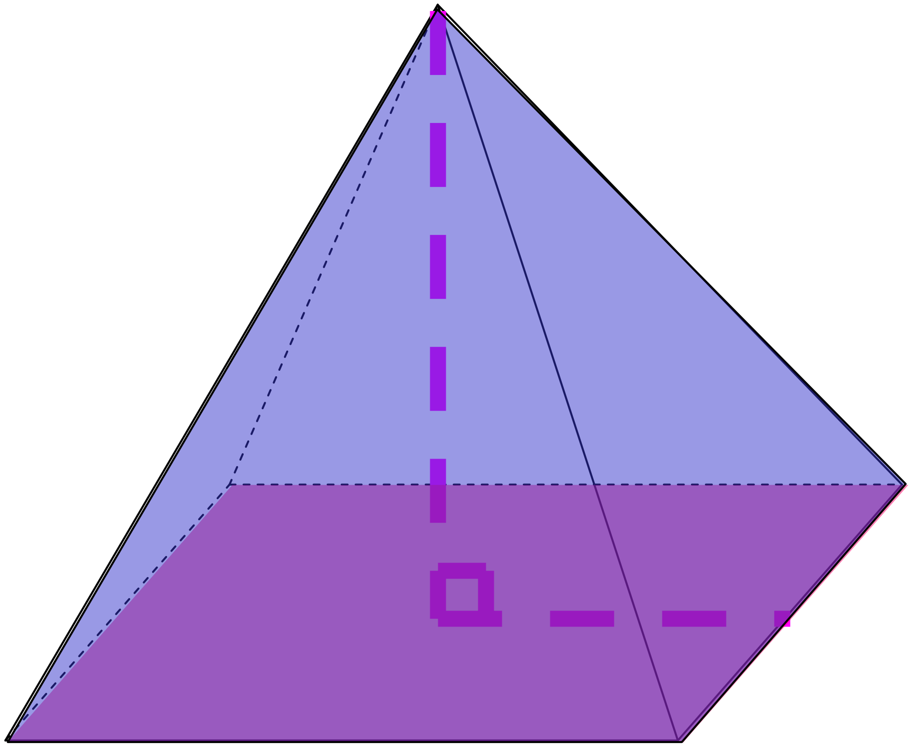
Slant Height (l)



Lateral Area = ½pl

★ Theorem 12-4

The volume of a pyramid equals one third the area of the base times the height of the pyramid.



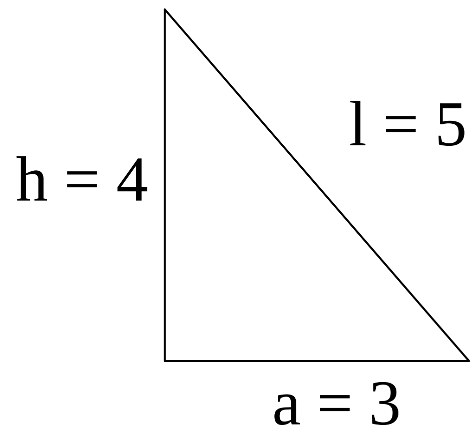
Base Area (B)

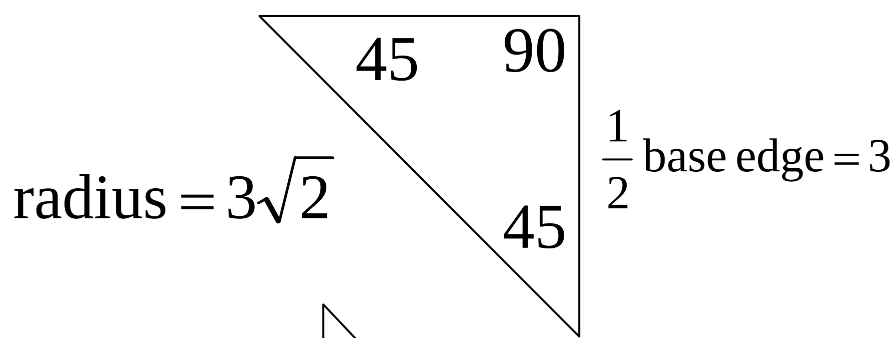
Height (h)

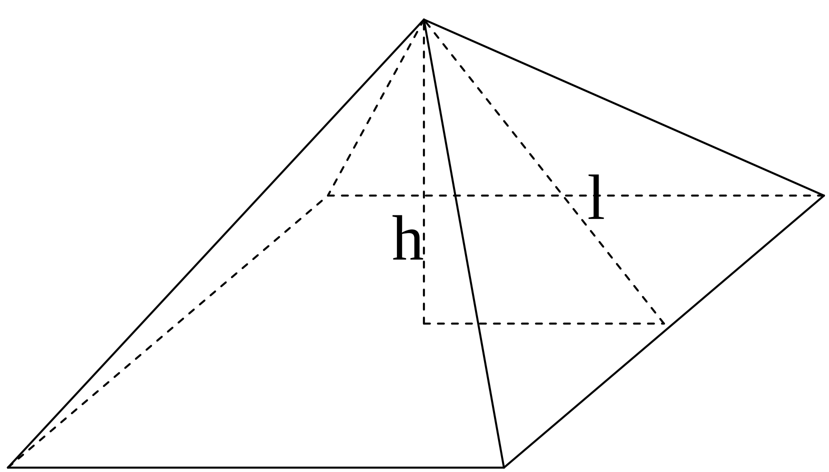
$$\text{Volume} = \frac{1}{3}Bh$$

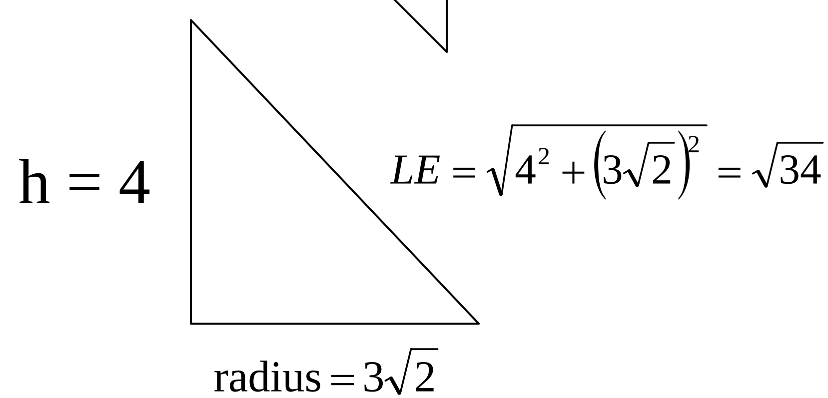
Sample Problems

regular square pyramid	1.
height, h	4
slant height, l	5
base edge	6
lateral edge	$\sqrt{34}$



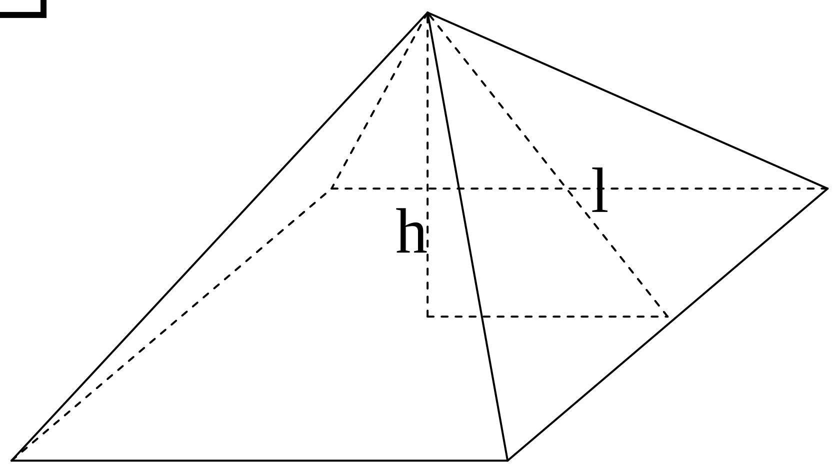






Sample Problems

regular square pyramid	3.	5.
height, h	24	
slant height, l		5
base edge	14	8
lateral edge		



Sample Problems

Find the lateral area of the pyramid described.

7. A regular triangular pyramid with base edge 4 and slant height 6.

9. A regular square pyramid with base edge 12 and lateral edge 10.

Find the lateral area, total area and volume of each square pyramid.

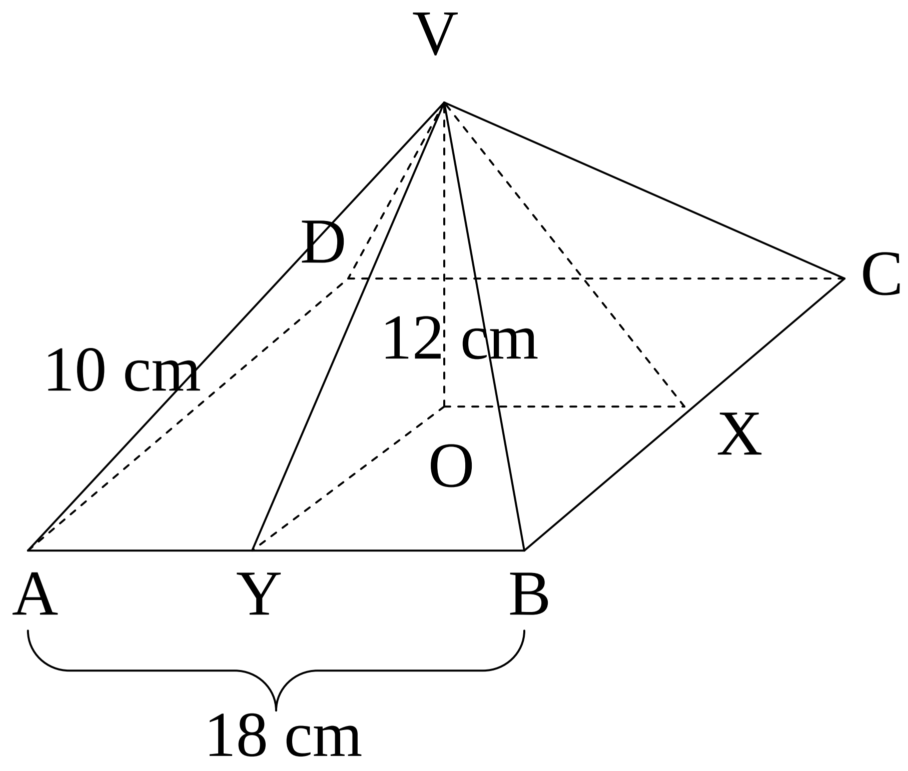
11. base edge = 6, height = 4

13. height = 12, slant height = 13

15. A pyramid has a base area of 16 and a volume of 32. Find its height.

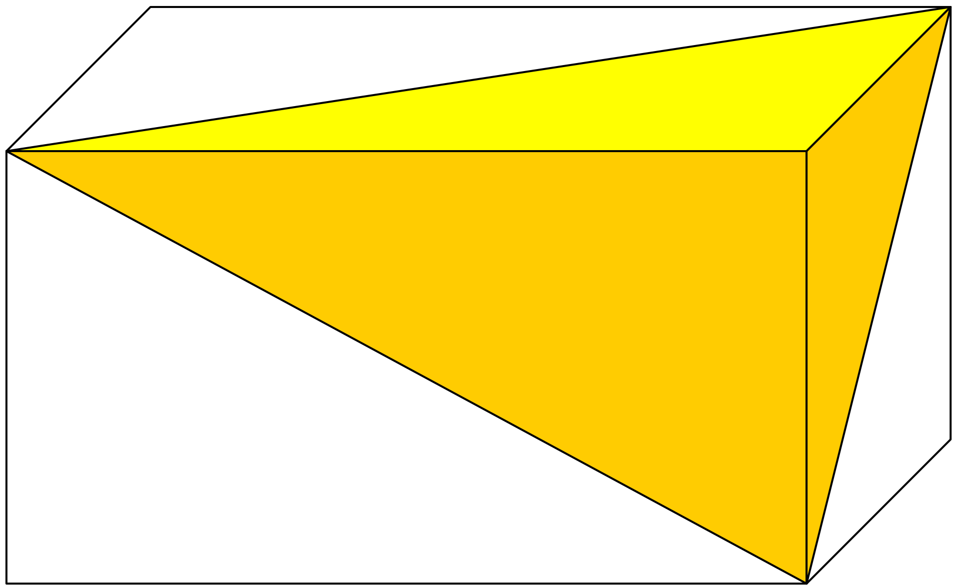
Sample Problems

17. VABCD is a pyramid with a rectangular bas 18 cm long by 10 cm wide. O is the center of the rectangle. The height, VO, of the pyramid is 12 cm. (a) Find VX and VY. (b) Find the lateral area of the pyramid. Why can't you use the formula?



Sample Problems

19. The shaded pyramid in the diagram is cut from a rectangular solid. How does the volume of the pyramid compare to the volume of the rectangular solid?



Sample Problems

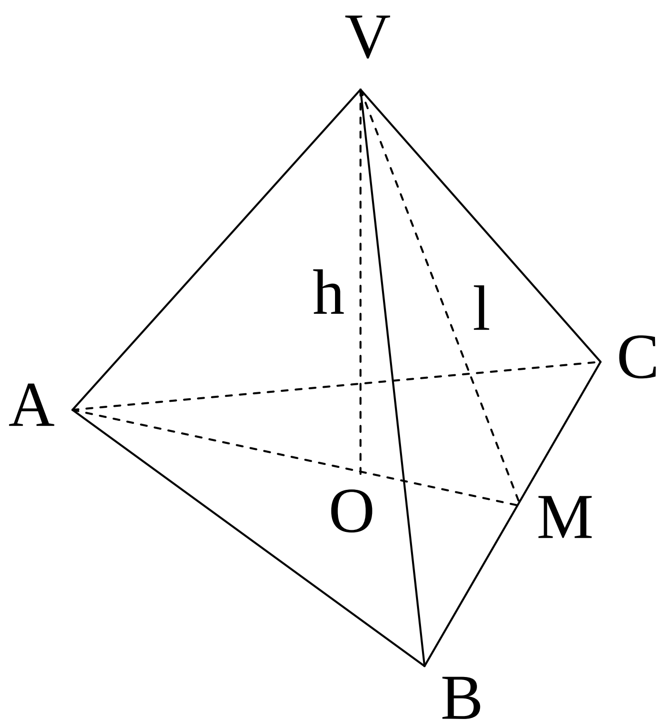
Given a regular triangular pyramid.

21. If AM = 9 & VA = 10 find h and l.

23a. If h = 4 and l = 5, find OM, OA and BC.

23b. Find the lateral area and volume.

25. If AB = 12 and VA = 10, find the lateral area and volume.



Section 12-3

Cylinders and Cones

Homework Pages 492-495:

2-32 evens

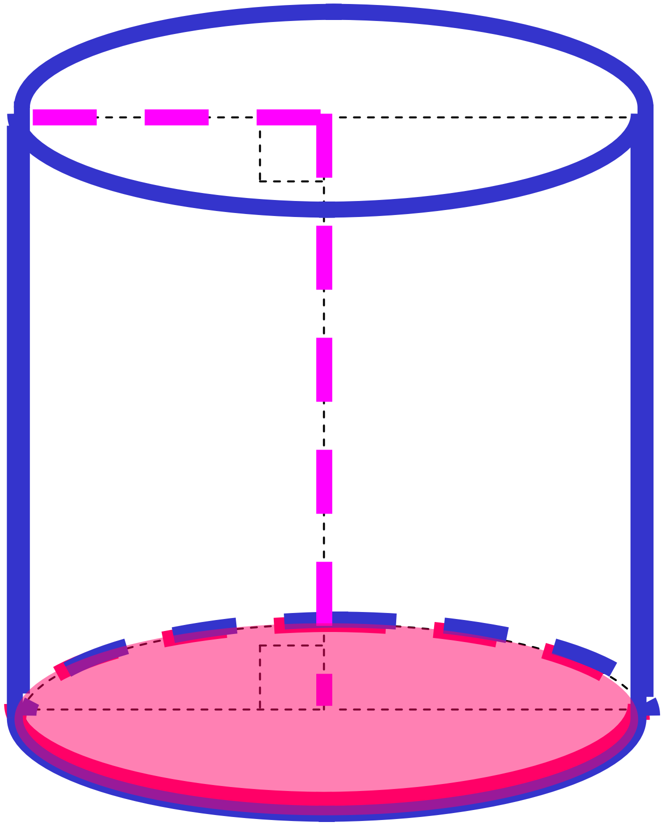
Excluding 4, 18, 20, 28

Objectives

A. Define and identify the parts of cylinders and cones.
B. Define and identify different types of cylinders and cones.
C. Define and calculate the lateral area of cylinders and cones.
D. Define and calculate the volume of cylinders and cones.
E. Understand and utilize the theorems of lateral area and volume of cylinders to solve real world problems.
F. Understand and utilize the theorems of lateral area and volume of cones to solve real world problems.

- cylinder: a prism with circular bases
- right cylinder: the segment joining the centers of the bases is an altitude
- cone: a pyramid with a circular base

Parts of a Cylinder



Base

Altitude

Lateral Face

Radius

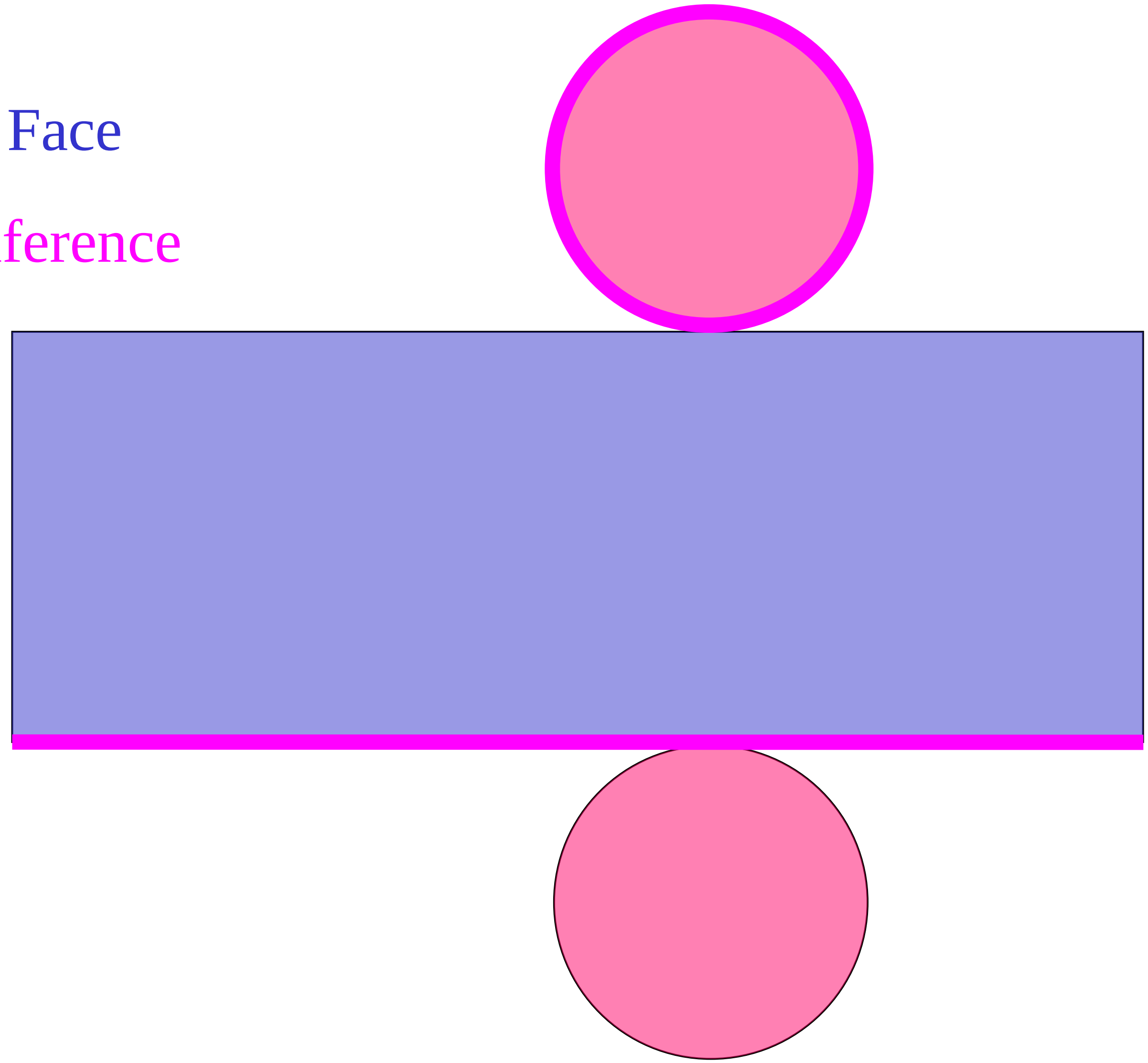
Base Area

Disassembled Right Cylinder

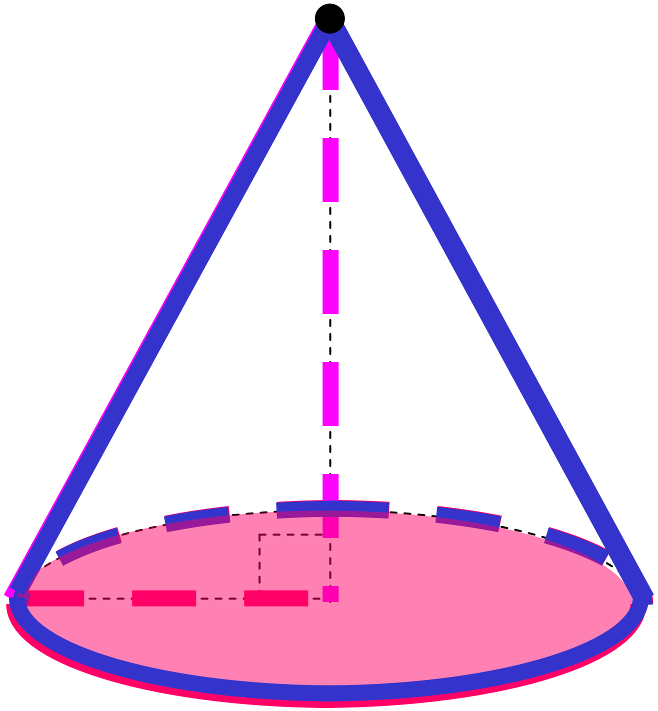
Bases

Lateral Face

Circumference



Parts of a Cone



Base

Altitude

Lateral Face

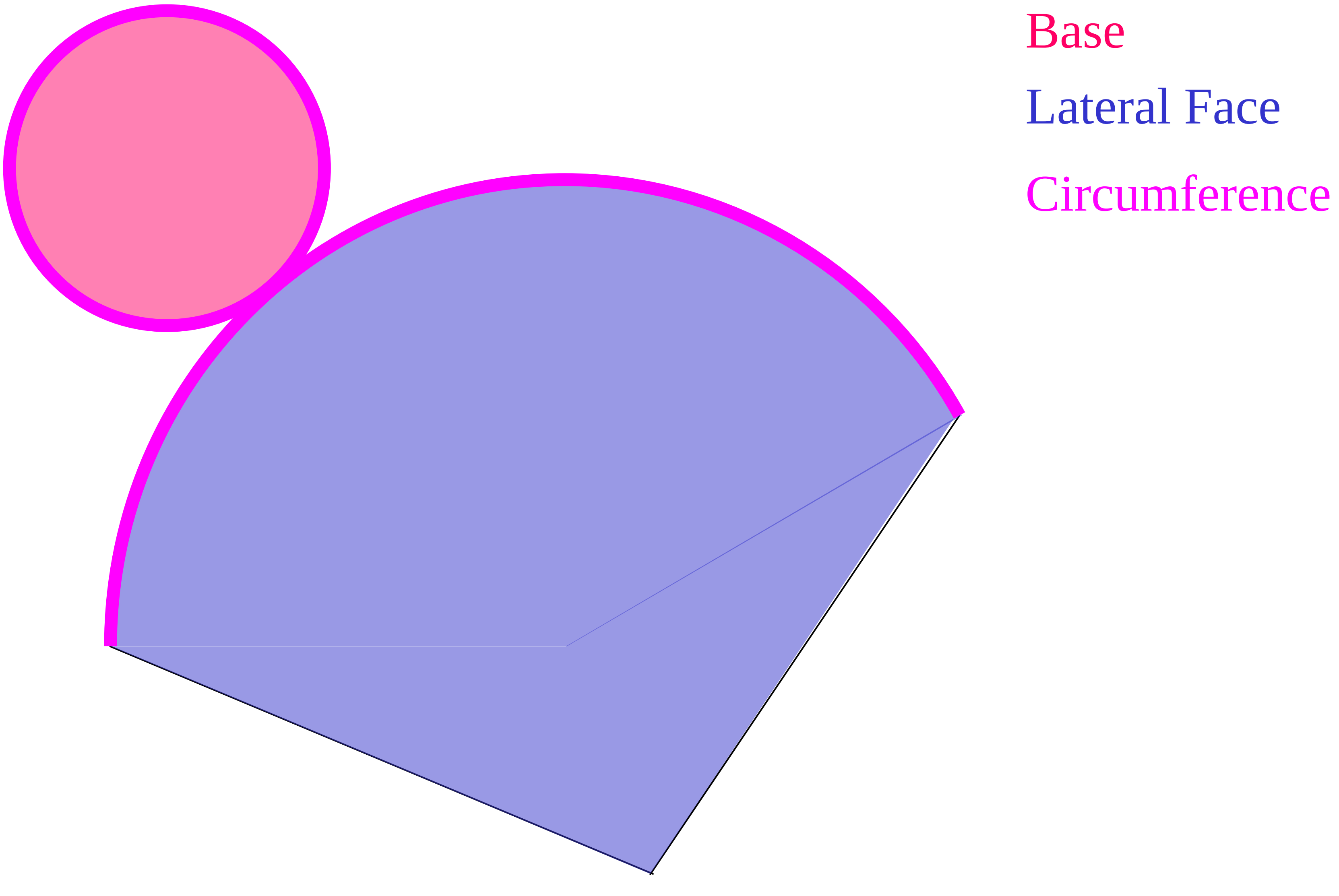
Radius

Slant Height

Base Area

Vertex

Disassembled Right Cone

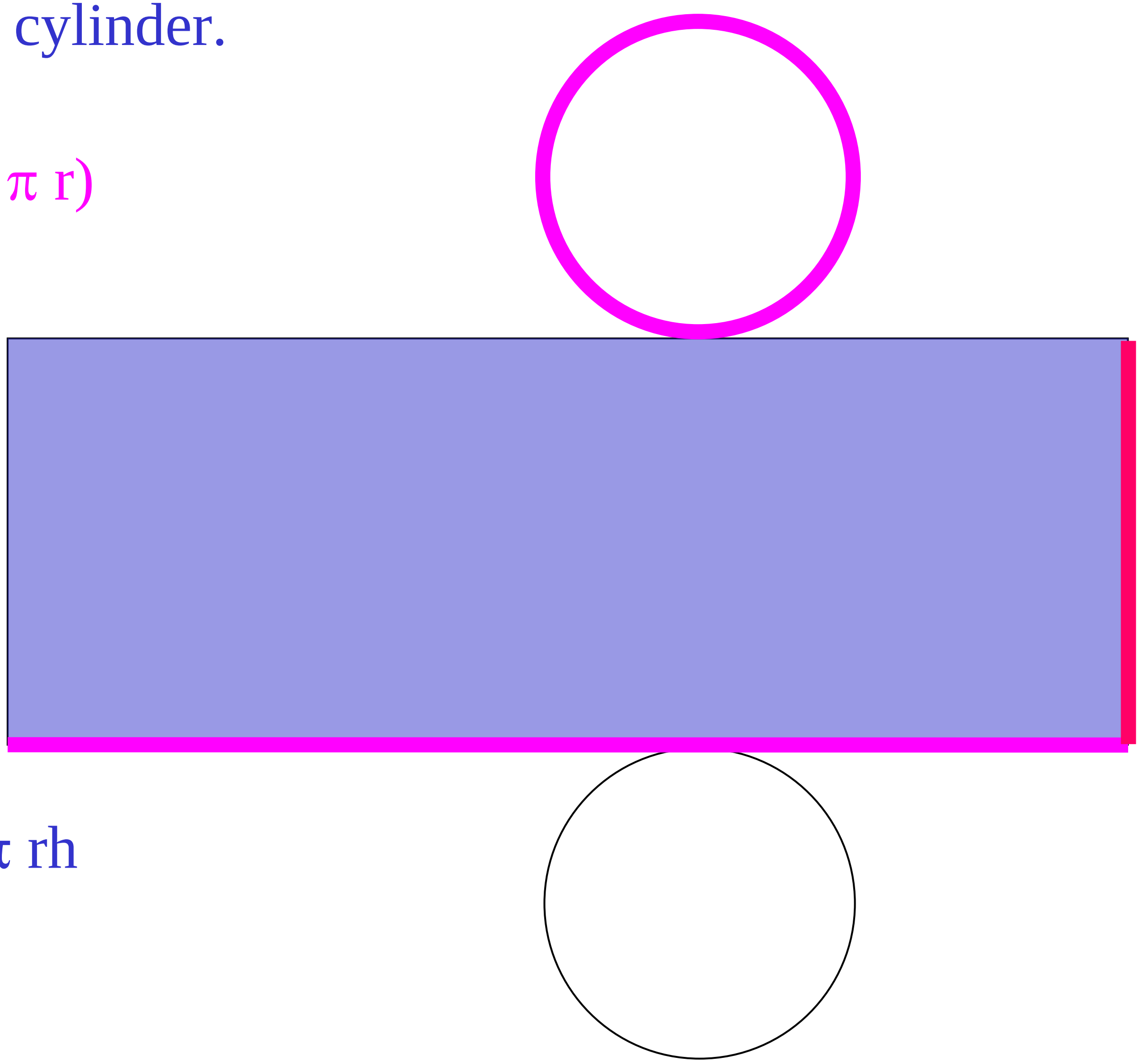


★ Theorem 12-5

The lateral area of a cylinder equals the circumference of a base times the height of the cylinder.

Circumference ($2 \pi r$)

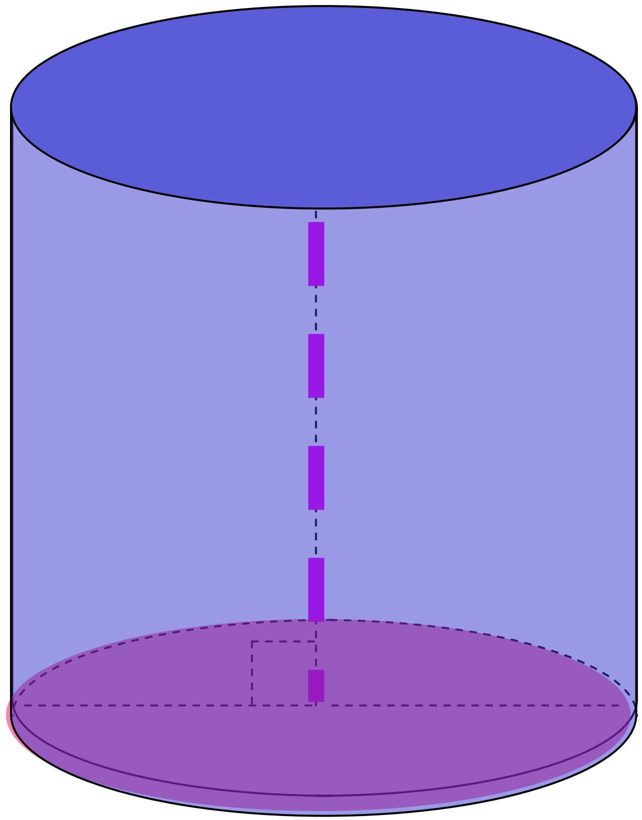
Height (h)



Lateral Area = $2 \pi rh$

★ Theorem 12-6

The volume of a cylinder equals the area of a base times the height of the cylinder.



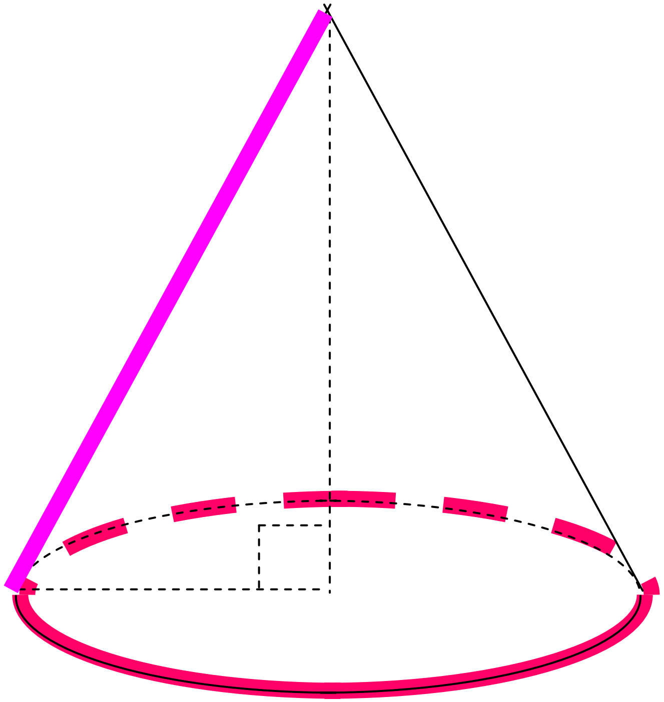
Base Area (πr^2)

Height (h)

Volume = $\pi r^2 h$

★ Theorem 12-7

The lateral area of a cone equals half the circumference of the base times the slant height.



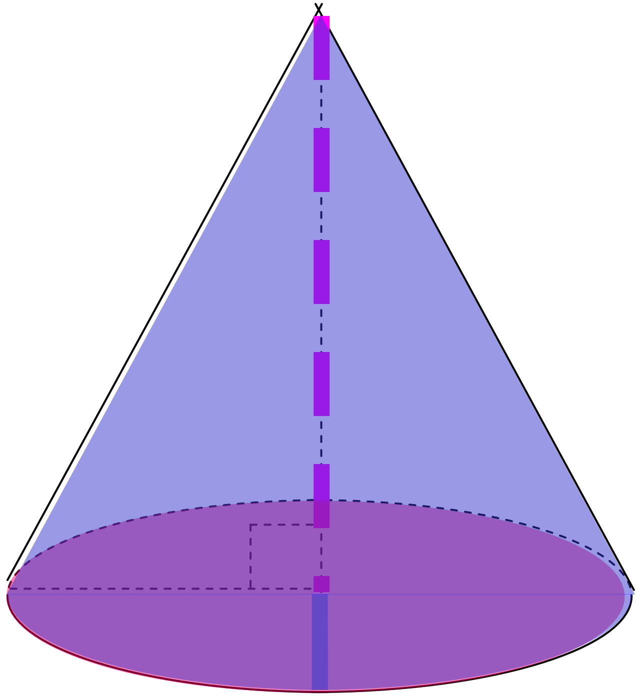
Circumference ($2\pi r$)

Slant Height

$$\text{Lateral Area} = \frac{1}{2}(2\pi rl) = \pi rl$$

★ Theorem 12-8

The volume of a cone equals one third the area of the base times the height of the cone.



Base Area (πr^2)

Height (h)

$$\text{Volume} = \frac{1}{3}\pi r^2 h$$

Sample Problems

Find the lateral area, total area and volume of each cylinder described.

1. $r = 4$; $h = 5$
3. $r = 4$; $h = 3$

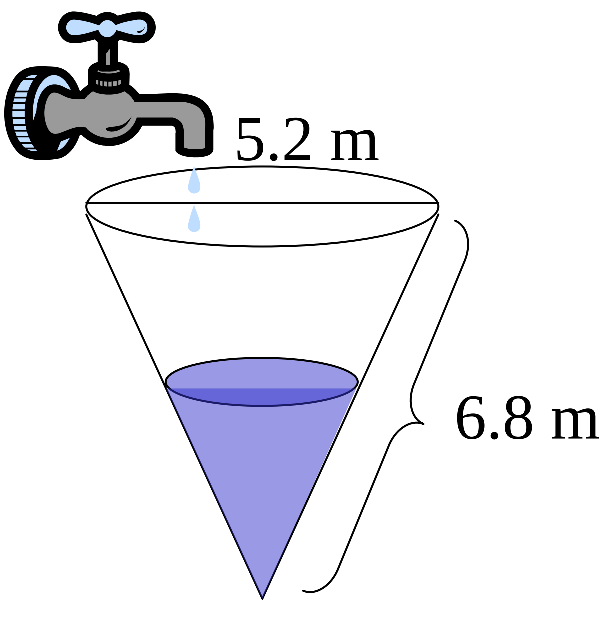
5. The volume of a cylinder is 64π. If $r = h$, find r.
7. The volume of a cylinder is 72π. If $h = 8$, find the lateral area.

Sample Problems

cone	r	h	l	L.A.	T.A.	V
9.	4	3				
11.	12		13			
13.			15	180π		
15.	15					600π

Sample Problems

17. The ratio of the radii of two cones is 1:2, and the ratio of the heights is also 1:2. What is the ratio of their (a) lateral areas (b) total areas and (c) volumes?
19. A cone and a cylinder both have a height 48 and radius 15. Give the ratio of their volumes.
21. A solid metal cylinder with a radius 6 and height 18 is melted down and recast as a solid cone with radius 9. Find the height of the cone.
23. Water is pouring into a conical reservoir at the rate of 1.8 m^3 per minute. Find to the nearest minute, the number of minutes it will take to fill the reservoir.



Sample Problems

25. The total area of a cylinder is 40π. If $h = 8$ find r.
27. In rectangle ABCD, AB = 10 and AD = 6. (a) The rectangle is rotated in space about AB. Describe the solid that is formed and find its volume. (b) Answer part (a) if the rectangle is rotated about AD.
31. An equilateral triangle with 6 cm sides is rotated about an altitude. Describe the solid formed and find its volume.
33. A regular square pyramid with base 4 is inscribed in a cone with height 6. What is the volume of the cone?
35. A cone is inscribed in a regular square pyramid with slant height 9 and base edge 6. Make a sketch. Then find the volume of the cone.

Section 12-4

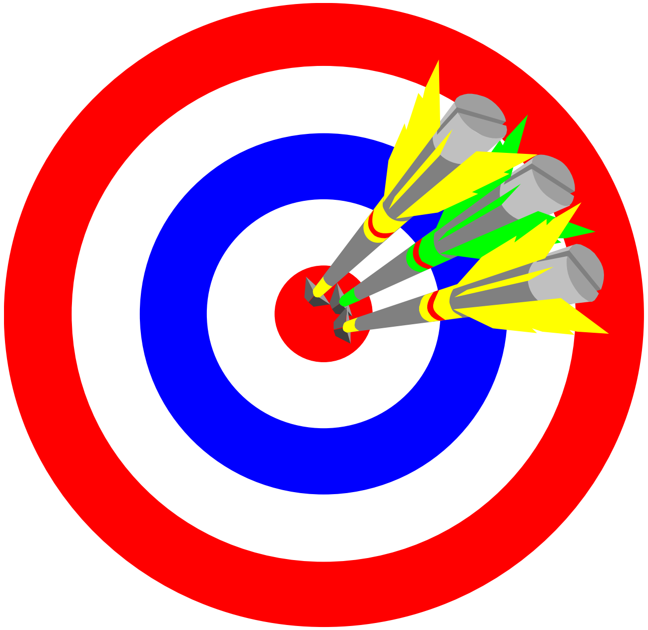
Spheres

Homework Pages 500-502:

2-30 evens

Excluding 4, 10, 16, 18, 28

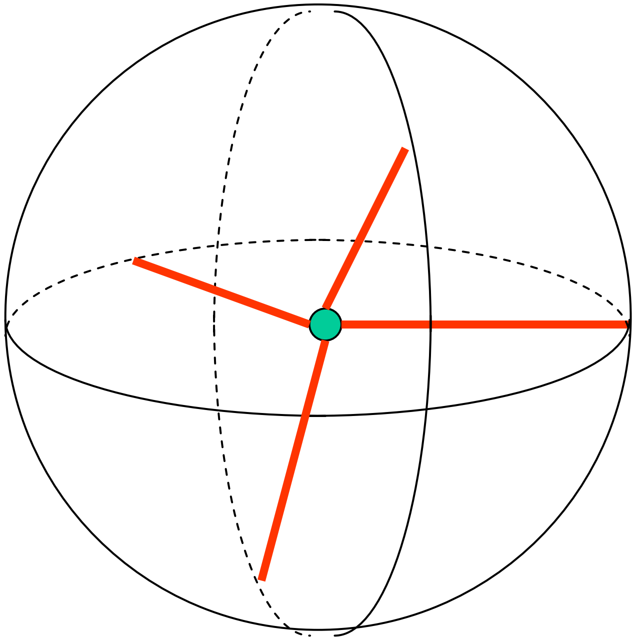
Objectives



A. Define and identify the parts of spheres.
B. Define and calculate the surface area of spheres.
C. Define and calculate the volume of spheres.
D. Understand and utilize the theorems of surface area and volume of spheres to solve real world problems.

Spheres

A sphere is the set of all points equidistant (radius) from a single fixed point (center).



Center

Radius

Sphere

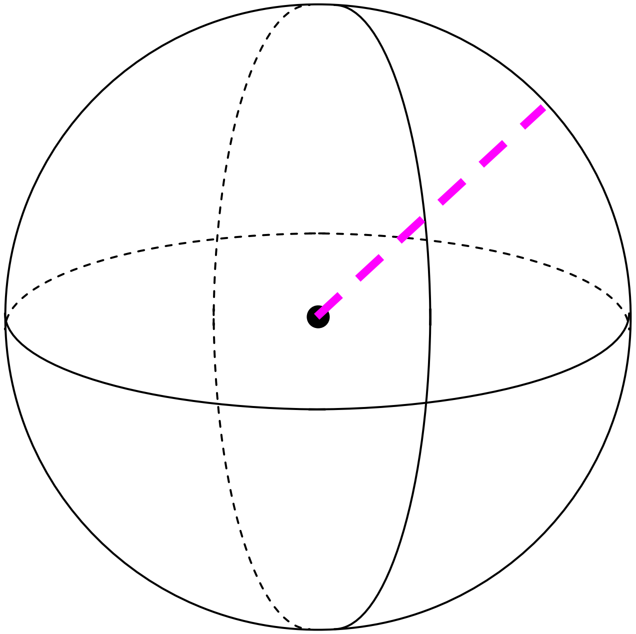
Radii

★ Theorem 12-9

The surface area of a sphere equals
4 π times the square of the radius.

Radius (r)

Area = $4\pi r^2$

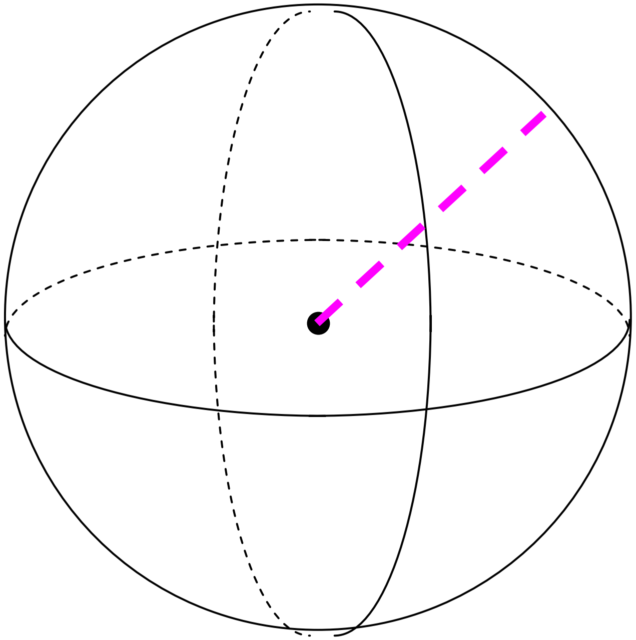


★ Theorem 12-10

The volume of a sphere equals four-thirds π times the cube of the radius.

Radius (r)

$$\text{Volume} = \frac{4}{3}\pi r^3$$



Sample Problems

sphere	3.
r	$\frac{1}{2}$
A	
V	

$$Area = 4\pi r^2$$

$$Area = 4\pi\left(\frac{1}{2}\right)^2 = 4\left(\frac{1}{4}\right)\pi$$

$$= \pi\ units^2$$

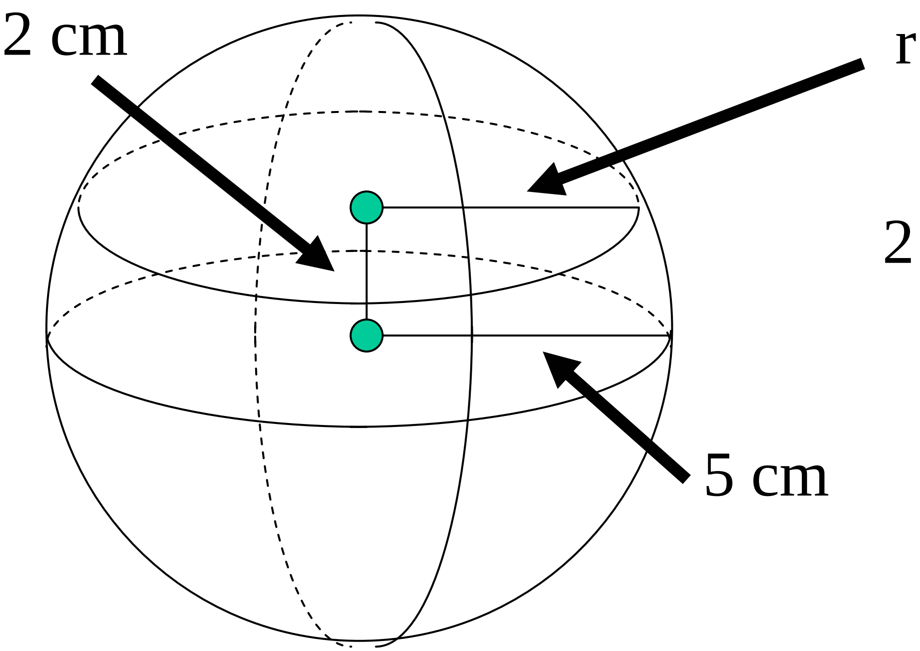
$$Volume = \frac{4}{3}\pi r^3$$

$$Volume = \frac{4}{3}\pi\left(\frac{1}{2}\right)^3 = \left(\frac{4}{3}\right)\left(\frac{1}{8}\right)\pi$$

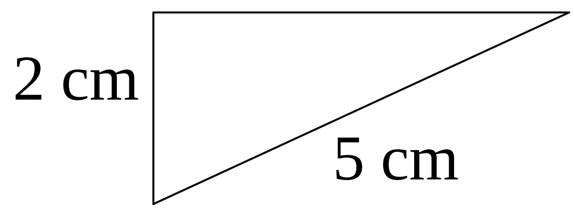
$$= \frac{1}{6}\pi = \frac{\pi}{6}\ units^3$$

Sample Problems

13. Find the area of a circle formed when a plane passes 2 cm from the center of a sphere with radius 5 cm.



$$r = \sqrt{5^2 - 2^2} = \sqrt{21}$$



$$Area = \pi r^2 = \pi\left(\sqrt{21}\right)^2 = 21\pi\ cm^2$$

Sample Problems

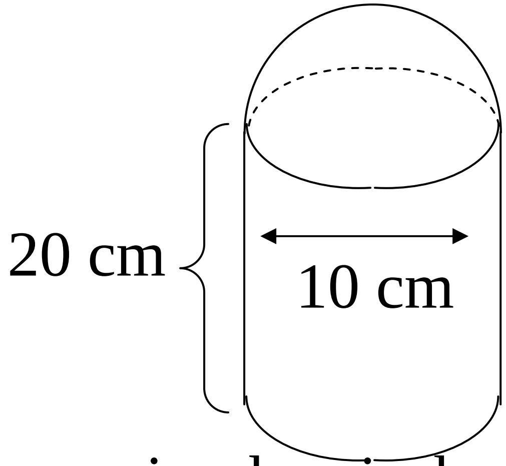
sphere	1.	5.	7.
r	7	$\frac{1}{2}$	$\sqrt{2}$
A		64π	
V			

Sample Problems

9. If the radius of a sphere is doubled, the area of the sphere is multiplied by ____ and the volume of the sphere is multiplied by ____?
11. The area of a sphere is π. Find the diameter of the sphere.
15. A sphere has a radius 2 and a hemisphere has radius 4. Compare the volumes.
17. Approximately 70% of the Earth's surface is covered by water. The radius of the Earth is approximately 6380 km. Find the amount of the Earth's surface covered by water to the nearest million square km.

Sample Problems

19. A silo of a barn consists of a cylinder capped by a hemisphere, as shown. Find the volume of the silo



21. An experimental one-room house is a hemisphere with a floor. If three cans of paint are required to cover the floor, how many cans of paint are needed to paint the ceiling?

23. A solid metal ball with a radius of 8 is melted down and recast as a solid cone with the same radius. (a) What is the height of the cone? (b) What is the ratio of the lateral area of the cone to the area of the sphere?

25. A sphere with radius r is inscribed in a cylinder. Find the volume of the cylinder in terms of r.

Sample Problems

29. A circle with diameter 9 is rotated about the diameter. Find the area and the volume of the solid formed.

Section 12-5

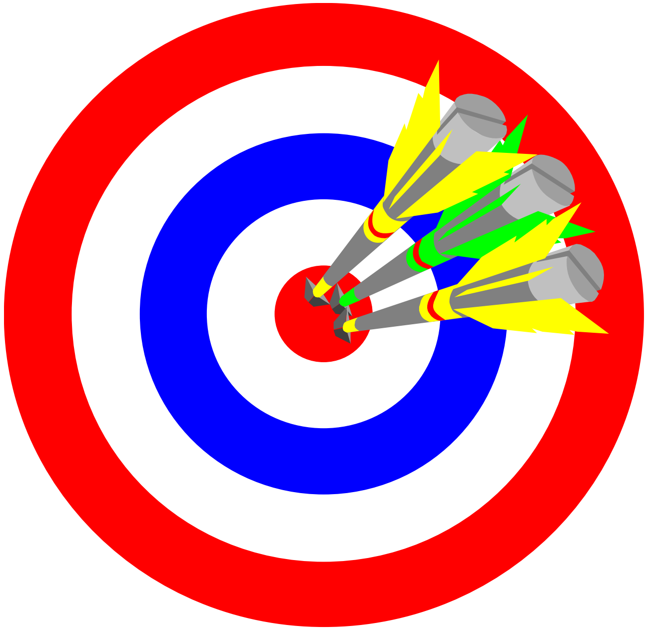
Areas and Volumes of Similar Solids

Homework Pages 511-512:

2-18 evens

Excluding 16

Objectives



A. Define and identify similar solids.

B. Determine the scale factor of similar solids.

C. Understand and utilize the relationships between the scale factor, ratio of perimeters, ratio of areas, and ratio of volumes.

D. Use the scale factor of similar solids to find sides, areas and volumes of similar solids.

E. Apply the concept of similar solids to real world problems.

Similar Solids

- ★ similar solids → Solids that have the same shape but not necessarily the same size.
  - The number of bases must be the same.
    - No bases → sphere
    - One base → pyramids and cones
    - Two bases → prisms and cylinders.
  - The types of bases must be the same.
    - No bases → sphere
    - Polygons (same # of sides) → prisms and pyramids.
    - Circles → cylinders and cones.

Similar Solids - continued

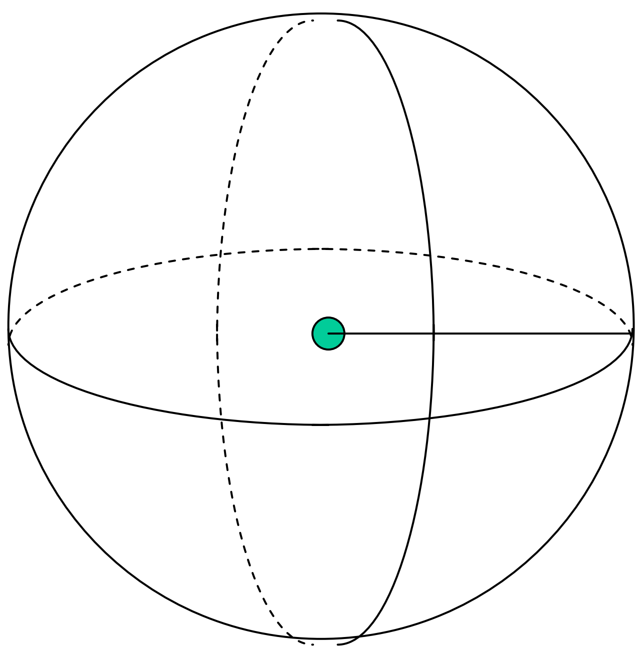
- Corresponding parts must be in proportion.
  - Prisms:
    - Base polygon of prism 1 must be similar to base polygon of prism 2 with a scale factor of a:b.
    - Height of prism 1 to height of prism 2 must have same ratio as scale factor (a:b).
  - Pyramids:
    - Base polygon of pyramid 1 must be similar to base polygon of pyramid 2 with a scale factor of a:b.
    - Altitude of pyramid 1 to altitude of pyramid 2 must have same ratio as scale factor (a:b).

Similar Solids - continued

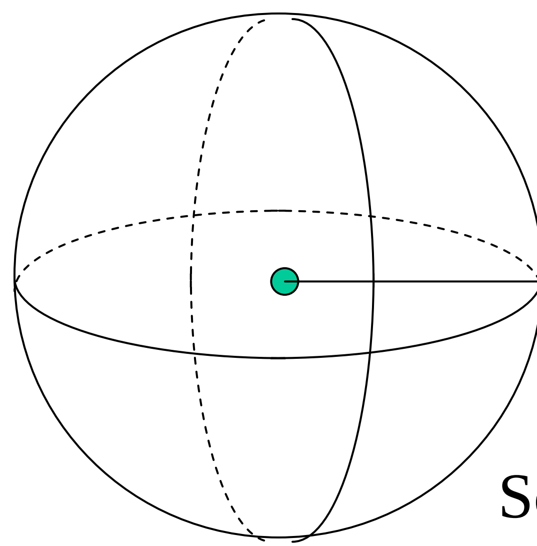
- Corresponding parts must be in proportion.
  - Cylinders:
    - Ratio of the radius of cylinder 1 to the radius of cylinder 2 must be the same as the ratio of the height of cylinder 1 to the height of cylinder 2.
    - Ratio of the radii or of the heights is the scale factor.
  - Cones:
    - Ratio of the radius of cone 1 to the radius of cone 2 must be the same as the ratio of the altitude of cone 1 to the altitude of cone 2.
    - Ratio of the radii or of the altitudes is the scale factor.

Similar Solids - continued

- Corresponding parts must be in proportion.
  - Spheres:
    - All spheres are similar to each other with the ratio of their radii as the scale factor.



Radius = 7



Radius = 5

Scale Factor = 7:5

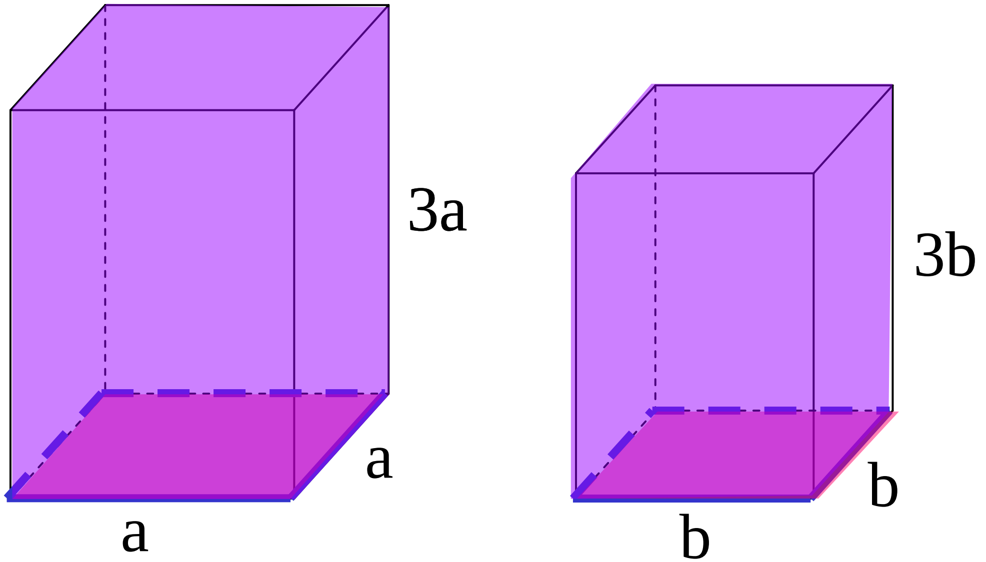
★ Theorem 12-11

If the scale factor of two similar solids is a : b, then:

(1) the ratio of corresponding perimeters is a : b

(2) the ratio of the base areas, of the lateral areas, and of the total areas is $a^2 : b^2$

(3) the ratio of the volumes is $a^3 : b^3$



$$\frac{4a}{4b} = \frac{a}{b}$$

$$\frac{(a)(a)}{(b)(b)} = \frac{a^2}{b^2}$$

$$\frac{(a)(a)(3a)}{(b)(b)(3b)} = \frac{3a^3}{3b^3} = \frac{a^3}{b^3}$$

Sample Problems

1. Two cones have radii 6 and 9. The heights are 10 and 15. Are the cones similar?

   Similar shapes? Both are cones with one circular base.

   Corresponding parts in proportion?
   Radius 1: radius 2 → 6:9 → 2:3
   Height 1: height 2 → 10:15 → 2:3
   All corresponding parts are in proportion.
   Scale factor 2:3.

Sample Problems

3. Two similar cylinders have radii 3 and 4. Find the ratio of: (a) heights (b) base circumferences (c) lateral areas (d) volumes

Scale factor = ?

Scale factor is ratio of corresponding parts.
Radius 1: radius 2 → 3:4
Scale factor → 3:4. Therefore a = 3, b = 4.

(a) Ratio of heights?

Ratio of corresponding parts (height to height) must = scale factor.
a = 3, b = 4. Therefore scale factor = 3:4

(b) Ratio of base circumferences?

Remember that a circumference is a perimeter of a circle.
From theorem 12-11, ratio of perimeters = scale factor.
3:4

Sample Problems

3. Two similar cylinders have radii 3 and 4. Find the ratio of: (a) heights (b) base circumferences (c) lateral areas (d) volumes

(c) Lateral Areas = ?

Scale factor = 3:4 with a = 3, b = 4.
From theorem 12-11, ratio of lateral areas = a^2:b^2.
3^2:4^2 → 9:16

(d) Ratio of volumes?

Scale factor = 3:4 with a = 3, b = 4.
From theorem 12-11, ratio of volumes = a^3:b^3.
3^3:4^3 → 27:64

Sample Problems

7. Two similar cones have volumes 8π and 27π. Find the ratios of: (a) the radii (b) the slant heights (c) the lateral areas.

Ratio of volumes → 8π:27π SIMPLIFIES TO → 8:27

Pattern for volume ratios → $a^3:b^3$

$a^3 = 8 : b^3 = 27$

$a = \sqrt[3]{8} : b = \sqrt[3]{27}$

$a = 2 : b = 3$

$\text{Scale Factor} = 2 : 3$

(a) Radii are parts → 2:3

(b) Slant heights are parts → 2:3

(c) Lateral areas are areas

Pattern → $a^2:b^2$ → $2^2:3^2$ → 4:9

Sample Problems

5. Assume the Earth and the moon are smooth spheres with diameters 12,800 km and 3,200 km. Find the ratios of: (a) lengths of their equators (b) areas (c) volumes

9. A model airplane states that the scale is 1:200. Find the ratio of the amounts of paint required to cover the model and the real plane.

Sample Problems

11. Two similar cones have radii of 4 and 6. The total area of the smaller cone is 36π. Find the total area of the larger cone.
13. Two balls made of the same metal have radii of 6 and 10. If the smaller ball weighs 4 kg, find the weight of the larger ball to the nearest 0.1 kg.
15. A certain kind of string is sold in a ball 6 cm in diameter and in a ball 12 cm diameter. The smaller ball costs $1 and the larger one costs $6.50. Which is the better buy?
17. Two similar pyramids have lateral areas of 8 and 18. If the volume of the smaller pyramid is 32, what is the volume of the larger pyramid?

Sample Problems

19. A plane parallel to the base of a cone divides the cone into two pieces. Find the ratio of:

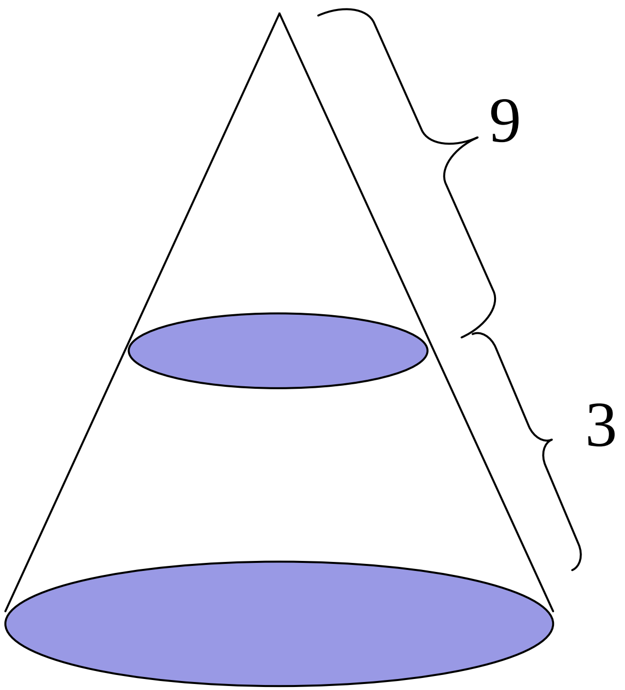
(a) the base areas

(b) lateral area of the top cone to the lateral area of the whole cone

(c) the lateral area of the top part of the cone to the lateral area of the bottom part of the cone

(d) the volume of the top cone to the volume of the whole cone

(e) the volume of the top part of the cone to the volume of the bottom part of the cone.



Sample Problems

21. A pyramid with height 15 cm is separated into two pieces by a plane parallel to base and 6 cm above it. What are the volumes of these two pieces if the volume of the original pyramid 250.

Chapter 12

Areas & Volumes of Solids

Review

Homework Page 519:

2-16 evens