Chapter 14

Transformations

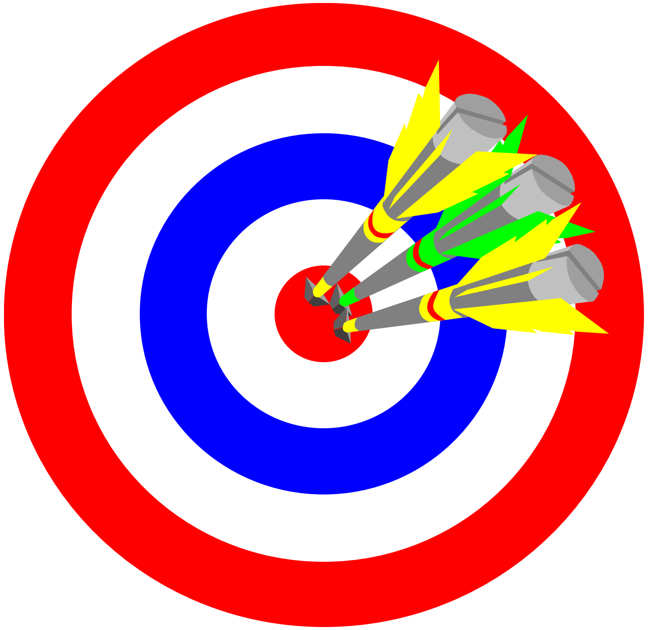
Questions For Thought

1. When you take a picture, how does the real world image become a reduced celluloid or digital image?
2. How are maps of the Earth made to scale?
3. How does a child learn to place a square block into a square hole?
4. How does a computer provide a ‘digital tour’ of a home?
5. How does computer aided design software all the use to rotate objects in 3-dimmensional space?
6. How does a video game designer make objects move across the screen in the direction and distance requested by the user?

The answer to each of these questions is the use of one or more types of **transformations**.

As we go through this chapter, see if you can identify which transformations are used to solve each of the above.

Objectives



A. Use the terms defined in the chapter correctly.
B. Properly use and interpret the symbols for the terms and concepts in this chapter.
C. Appropriately apply the theorems and corollaries in this chapter.
D. Correctly translate, glide reflect, reflect, rotate and dilate points on a graph.
E. Correctly find the composites and inverses of a mapping.
F. Correctly identify symmetries in a diagram.

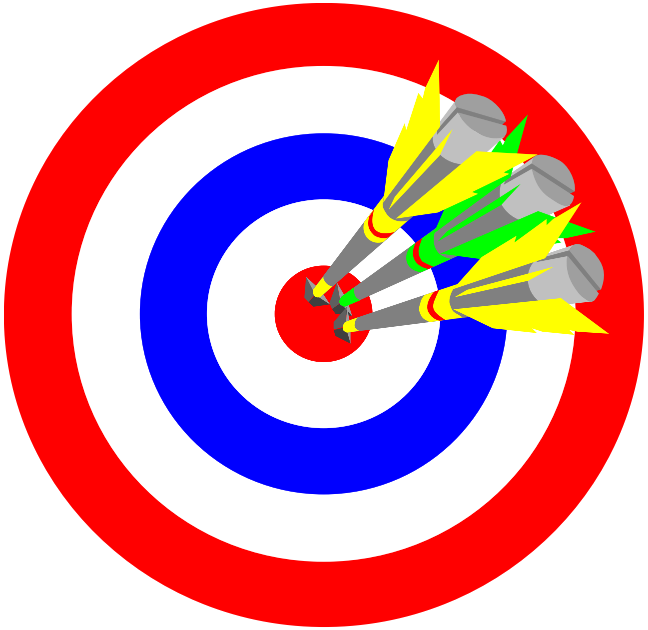
Section 14-0

Review of Functions

Handout

Do all problems on handout.

Objectives



A. Properly use the term function.
B. Identify mathematical functions.
C. Solve mathematical functions.

Functional Definition

Understanding the algebraic concept of functions will be critical to your understanding of transformations.

Definitions of the term function:

1. A function is a correspondence between two sets, D and R, that assigns each member of D exactly one member of R.
2. In a two variable equation, a function is such that for every input (x-value), there is exactly one output (y-value).
3. A function is a mathematical equation such that for every input in the domain there is exactly one output in the range.

Functional Forms

Unfortunately, there are multiple valid methods for describing a mathematical function.

- Functional Notation: $f(x) = x + 2$
  - The function f of x equals x + 2.
- Mapping Notation: $g: x \rightarrow 2x$
  - The function g takes x as an input and produces 2x for an output.
- Equation Notation: $y = 3x - 1$
  - ‘Disguised’ function which takes x as an input and produces 3x – 1 as an output.
- Ordered Pair Notation: (0,1), (2,3), (4,7)
  - First value in the ordered pair represent input
  - Second value in the ordered pair represent output

Checking if you have a function

There are several methods you can use to determine if an equation is truly a function.

The two most common are:

1. For every X-value in the domain, there is EXACTLY ONE Y-value in the range.
2. Graph the solution set for the equation. Use the vertical line test to determine if the equation is a function.

Checking if you have a function

1. For every X-value in the domain, there is EXACTLY ONE Y-value in the range.

Which sets of ordered (x, y) pairs represent a function?

Ordered Pairs Set A

$$(3,4),(4,3),(3,5),(5,3)$$

Ordered Pairs Set B

$$(3,4),(4,5),(5,6),(6,7)$$

Ordered Pairs Set C

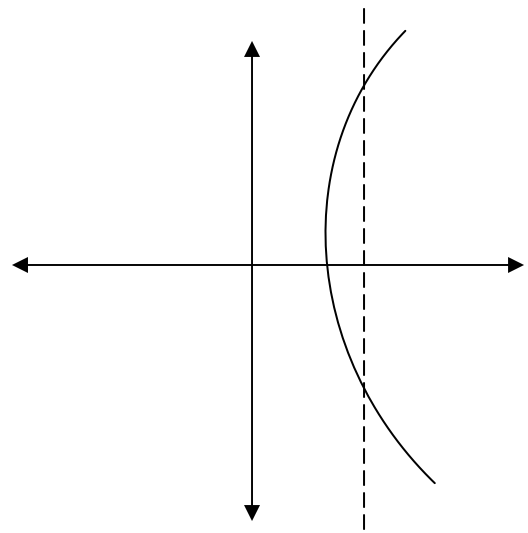
$$(3,4),(4,4),(5,4),(6,4)$$

Checking if you have a function

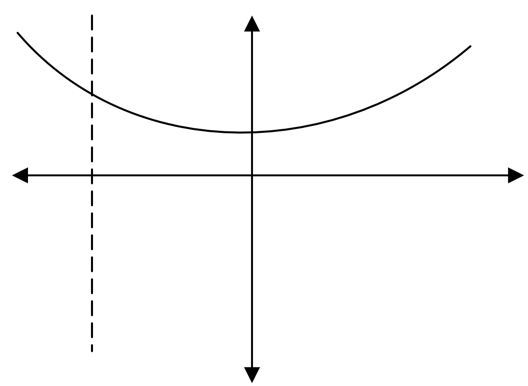
2. Graph the solution set for the equation. Use the vertical line test to determine if the equation is a function.

Which of these equations represent a function?

$$y^2 = x + 1$$



$$y = x^2 + 1$$



Evaluating a Function at a Point

The phrase 'evaluating a function at a point' basically means that you are given an input, you plug that input into the function, and the function produces an output.

Evaluate the function f(x) = 3x + 4 at 7.

$f(x) = 3x + 4$ where $x = 7$.

$f(7) = (3)(7) + 4 = 25$

25 is the evaluation (value) of the function at 7.

Domain and Range of a Function

Domain → All valid inputs to a function

- Can be affected by the nature of the function.
- Can be affected by the definition of the function.

Range → All possible outputs based on the Domain.

- Affected solely by the Domain of the function.

Determine the Domain and Range of Each Function

$$f(x) = 3x + 1$$

Domain = {All Real Numbers}
Range = {All Real Numbers}

$$f(x) = \frac{5}{x}$$

Domain = {All Real Numbers except x = 0}
Range = {All Real Numbers except y = 0}
Domain limited by nature of function.

$$f(x) = \sqrt{x - 2}$$

Remember a square root of a negative number is not a real number!

$$x - 2 \geq 0$$

therefore

$$x \geq 2$$

Domain = {All Real Numbers such that x ≥ 2}
Range = {All Real Numbers such that y ≥ 0}
Domain is limited by the nature of the function.

Determine the Domain and Range of Each Function

$$f(x) = 2x - 5 \text{ where } x \leq 3$$

Domain = {All Real Numbers such that $x \leq 3$}
Domain limited by the definition of the function.

Range is limited by the domain.
"Largest" value in the range, in THIS case, must be where $x = 3$.
$f(3) = (2)(3) - 5 = 1$
Range = {All Real Numbers such that $y \leq 1$}

Section 14-1

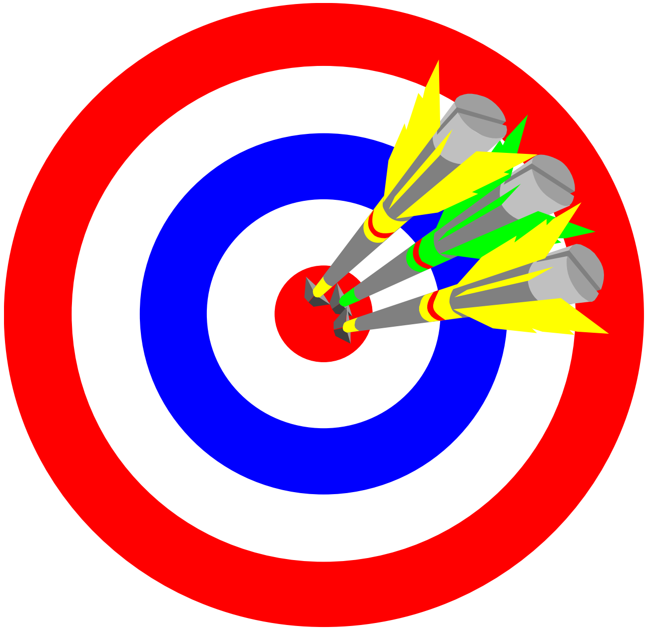
Mappings and Functions

Homework Pages 574-576:

2-20 evens

Excluding 14

Objectives



A. Understand and apply the terms preimage, mapping, one-to-one mapping, transformation, isometry, and congruence mapping.

B. Understand the relationship between algebraic functions and geometric mappings.

Mappings → The Basics

- ★ Mapping: a correspondence between a set of points such that for any value of the preimage there exists only one image, written
  - Name: (coordinates of preimage) → (coordinates of the image)
- Preimage:
  - The original or starting position of the figure
  - The ‘input’ to the mapping
- Image:
  - The copy or final position of the figure
  - The ‘output’ of the mapping

Functions → Reminder

Function: correspondence between sets of numbers such that for any value of the domain there exists only one value of the range.

Relation: correspondence between sets of numbers

Synergy between Functions and Mappings

Let $f(x)$ be a function such that $f(x) = x + 1$.

Let g be a mapping such that g maps a number to a value 'up one unit' on a coordinate plane.

Both $f(x)$ and g require an input value:

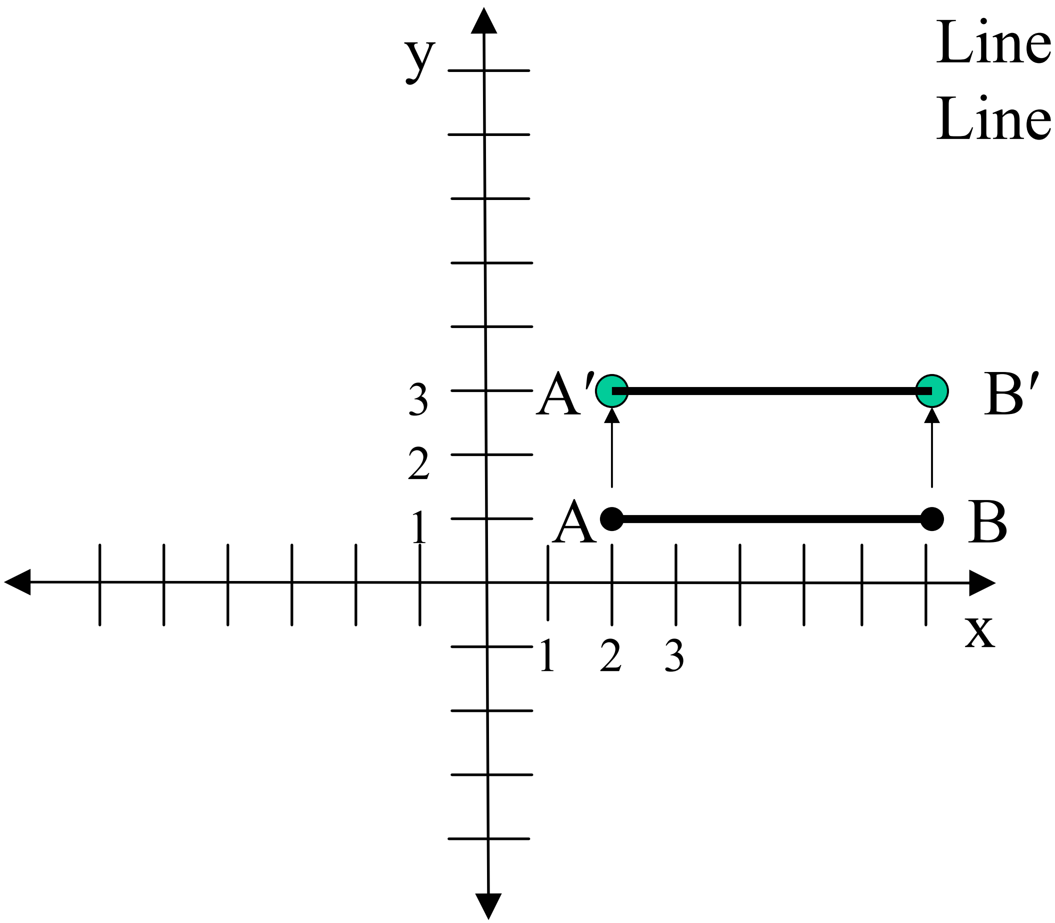
- $f(x)$ requires a real number input of x.
- g requires a coordinate value input of (x, y).

Both $f(x)$ and g produce an output value based on the input:

- $f(x)$ produces a real number based on the x input.
- g produces another coordinate (x, y) based on the original input coordinate (x', y').

Mapping Example

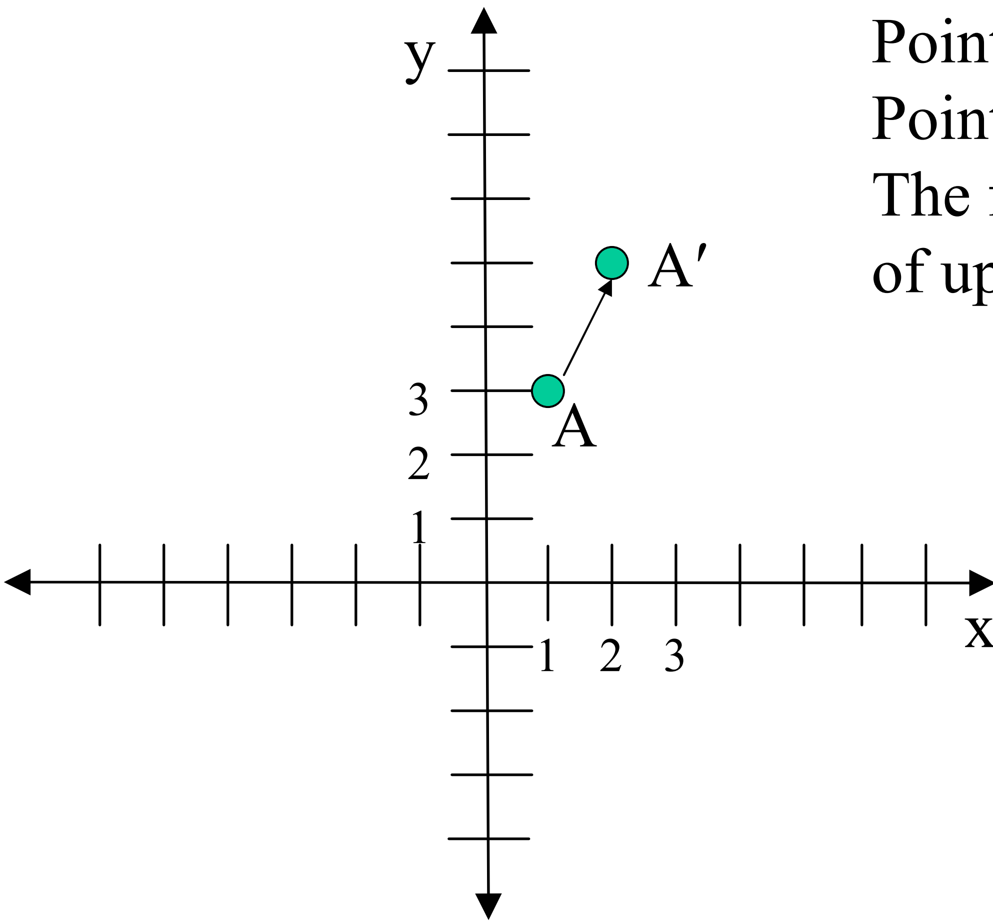
On the given coordinate plane, map each point on the line segment AB to a point up two units.



Line segment AB is the preimage.
Line segment A′B′ is the image.

Synergy?

Graph the function $f(x) = 2x + 1$.



Point A is similar to the preimage.
Point A′ is similar to the image.
The function is similar to a mapping of up two units and right one unit.

‘Types’ of Mappings

There are several special ‘types’ of mappings:

- One-to-one
- Transformations
- Isometry or Congruence

Mapping Types: One-to-one Mapping

One-to-one mapping: every value of the image has exactly one preimage.

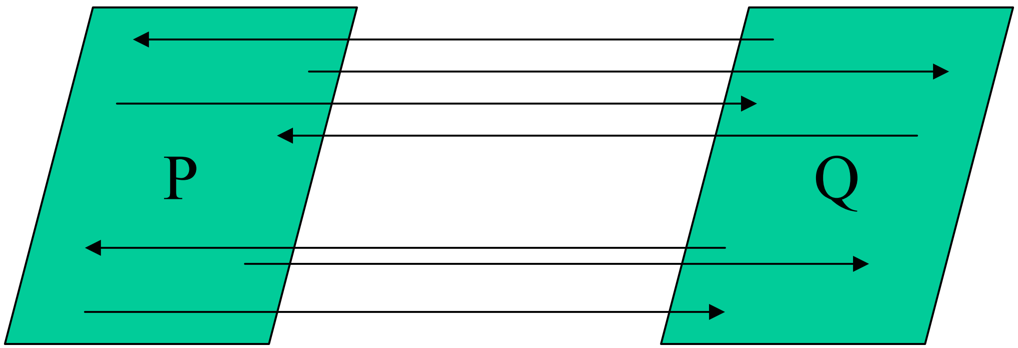
Remember!

- Preimage:
  - The original or starting position of the figure
  - The ‘input’ to the mapping
- Image:
  - The copy or final position of the figure
  - The ‘output’ of the mapping

One-to-one function: for every input to the function there is exactly one output and for every output of the function there is exactly one input.

Mapping Types: Transformation

Transformation: one-to-one mapping of the whole plane

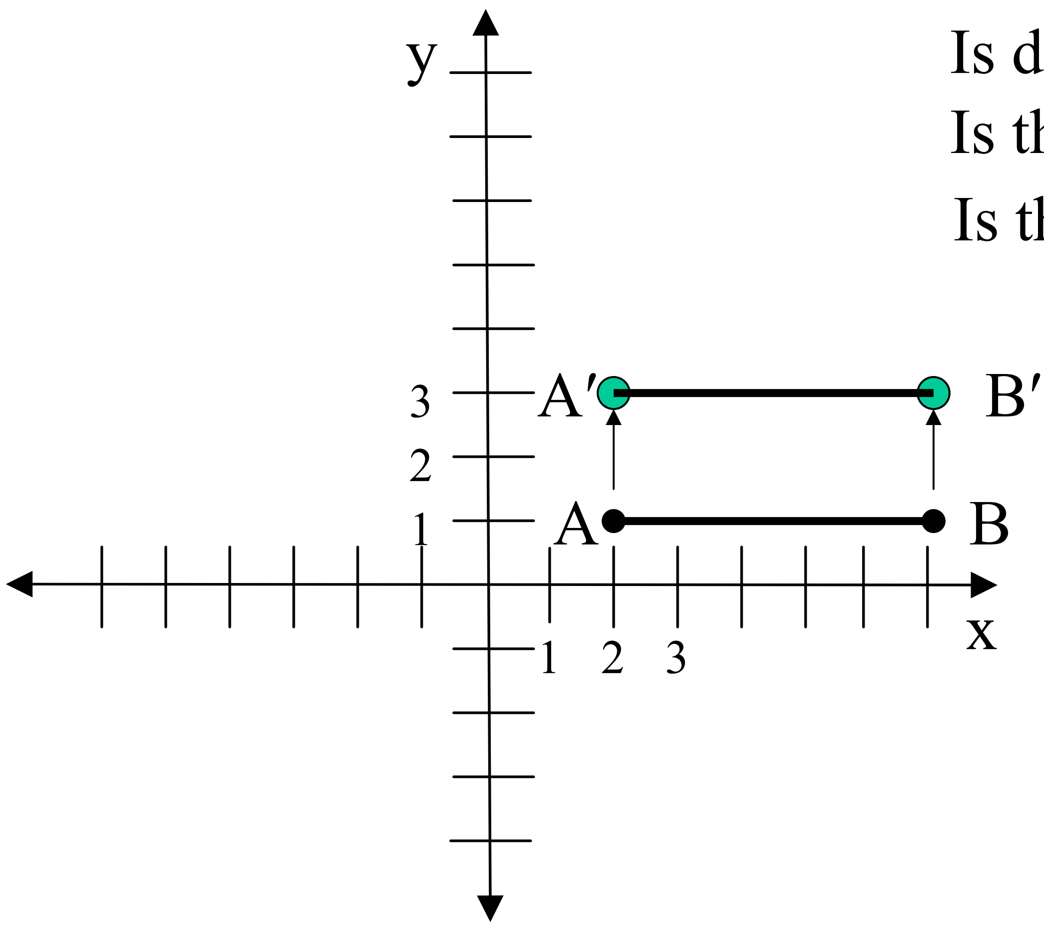


Every point on Plane P is mapped to exactly one point on Plane Q.

Likewise, every point on Plane Q is mapped to exactly one point on Plane P.

Mapping Types: Isometry

- Isometry (congruence mapping): a transformation that preserves distance



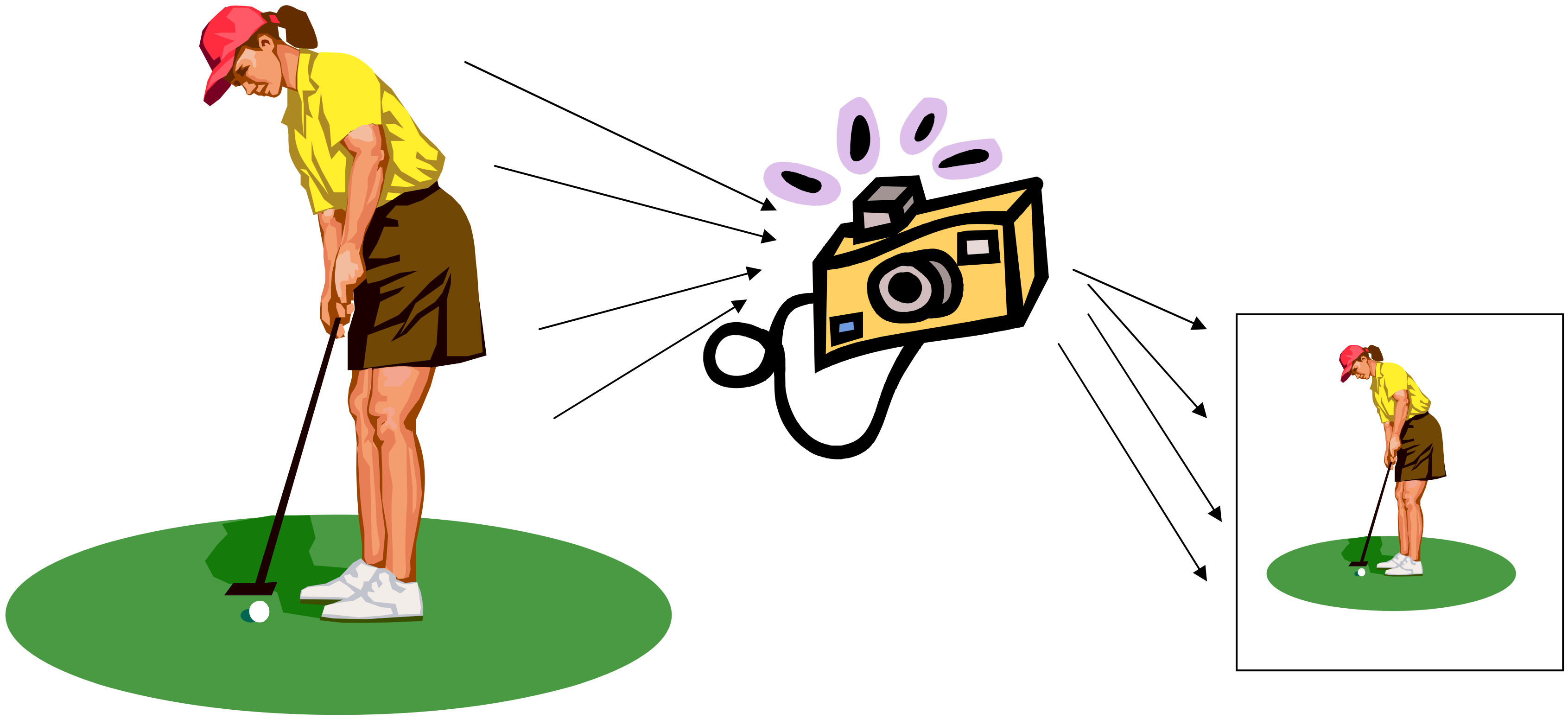
Is this a transformation?

Is distance preserved?

Is this an isometry?

Is this a congruence mapping?

Give an example of something that is a transformation but is NOT an isometry.



Theorem 14-1

An isometry maps a triangle to a congruent triangle.

If isometry M: $\Delta ABC \rightarrow \Delta A'B'C'$ then $\Delta ABC \cong \Delta A'B'C'$

Theorem 14-1 Corollary 1

An isometry maps an angle to a congruent angle.

If isometry M: $\angle A \to \angle A'$ then $\angle A \cong \angle A'$

Theorem 14-1 Corollary 2

An isometry maps a polygon to a polygon with the same area.

If isometry M: POLYGON $\rightarrow$ P′O′L′Y′G′O′ N′ then
Area of POLYGON = Area of P′O′L′Y′G′O′ N′

Sample Problems

1. If function $f{:}x \rightarrow 5x - 7$, find the image of 8 and the preimage of 13.

Remember:

- the preimage is similar to the 'input' of a function.
- the image is similar to the 'output' of a function.

Therefore, you could 'rewrite' $f{:}x \rightarrow 5x - 7$ as $f(x) = 5x - 7$.
If you are asked to find the image of 8, it is indicating that the 8 is the preimage, or the input.
Therefore $f(8) = (5)(8) - 7 = 33$.
Therefore, 33 is the image of 8.
Likewise, if you are asked to find the preimage of 13, it is indicating that 13 is the image, or the output.
$13 = 5x - 7$. Therefore, $x = 4$ and 4 is the preimage of 13.

Sample Problems

For each transformation in excercises 5 -10:

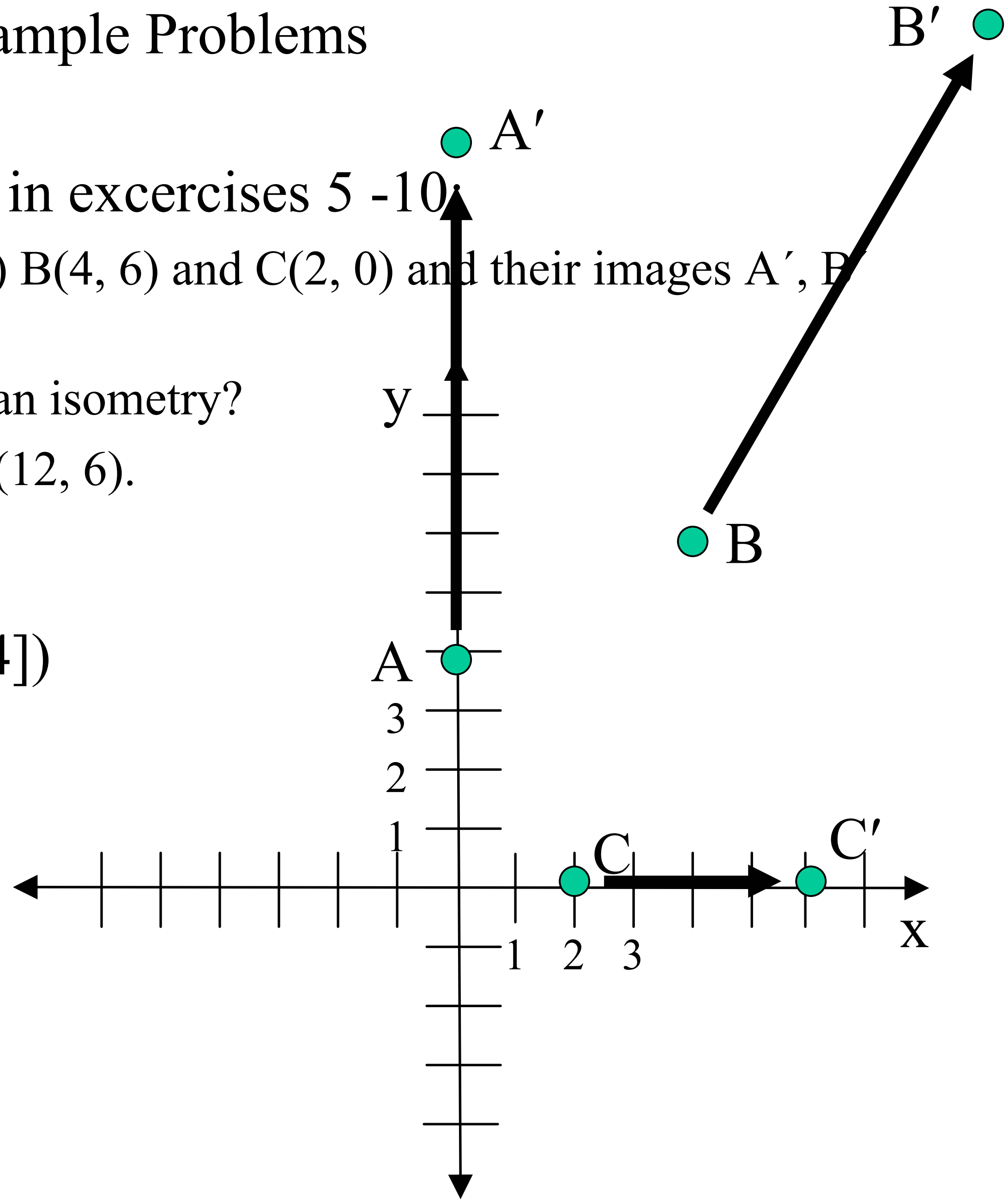
- Plot the points A(0, 4) B(4, 6) and C(2, 0) and their images A′, B′ and C′.
- Is the transformation an isometry?
- Find the preimage of (12, 6).

7. D: $(x, y) \to (3x, 3y)$

D: (0, 4) → (3[0], 3[4])
D: (0, 4) → (0, 12)

D: (4, 6) → (3[4], 3[6])
D: (4, 6) → (12, 18)

D: (2, 0) → (3[2], 3[0])
D: (2, 0) → (6, 0)



Sample Problems

3. If $f(x) = x^2 + 1$, find f(3) and f(- 3). Is f a one-to-one function?

For each transformation in excercises 5 -10:

– Plot the points A(0, 4) B(4, 6) and C(2, 0) and their images A′, B′ and C′.

– Is the transformation an isometry?

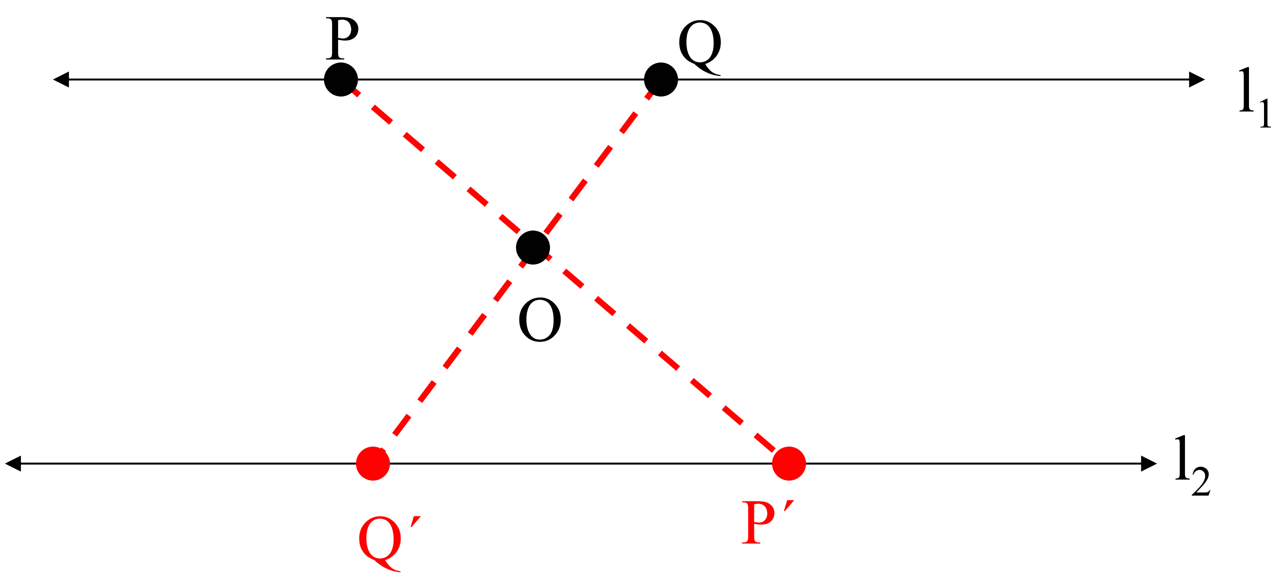
– Find the preimage of (12, 6).

5. T: $(x, y) \rightarrow (x + 4, y - 2)$

9. M: $(x, y) \rightarrow (12 - x, y)$

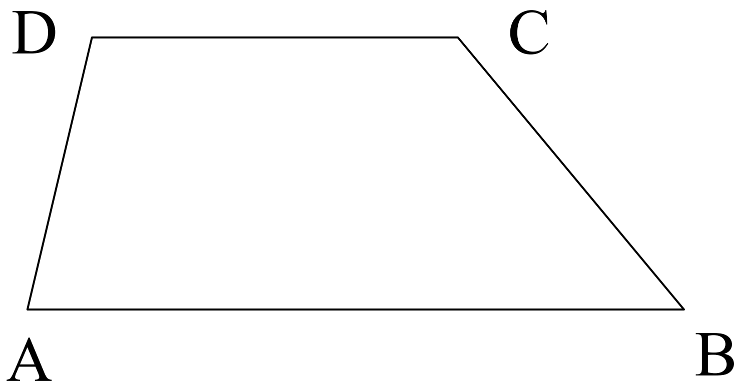
Sample Problems

11. O is a point equidistant from parallel line l_1 and l_2. A mapping M maps each point P of l_1 to the point P′ where $\overrightarrow{PO}$ intersects l_2. Is the mapping one-to-one? Does the mapping preserve or distort distance? If the lines were not parallel would the mapping preserve distance?



Sample Problems

13. ABCD is a trapezoid. Describe a way of mapping each point of DC to a point of AB so that the mapping is one-to-one. Is your mapping an isometry?



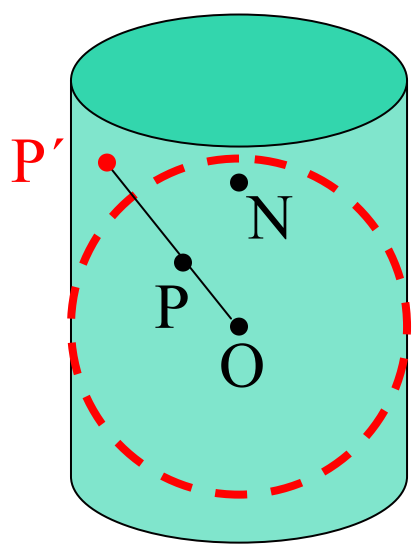
15. The transformation T: $(x, y) \rightarrow (x + y, y)$ preserves the area of figures even though it does not preserve distances. Illustrate this by drawing a square with vertices A(2, 3) B(4, 3) C(4, 5) and D(2, 5) and its image A′B′C′D′. Find the area and perimeter of each figure.

Sample Problems

A piece of paper is wrapped around a globe of the Earth to form a cylinder. O is the center of the Earth and a point P of the globe is projected along $\overrightarrow{OP}$ to a point P′ of the cylinder.

17. Is the image of the Arctic Circle congruent to the image of the equator?

19. Does the North Pole have an image?



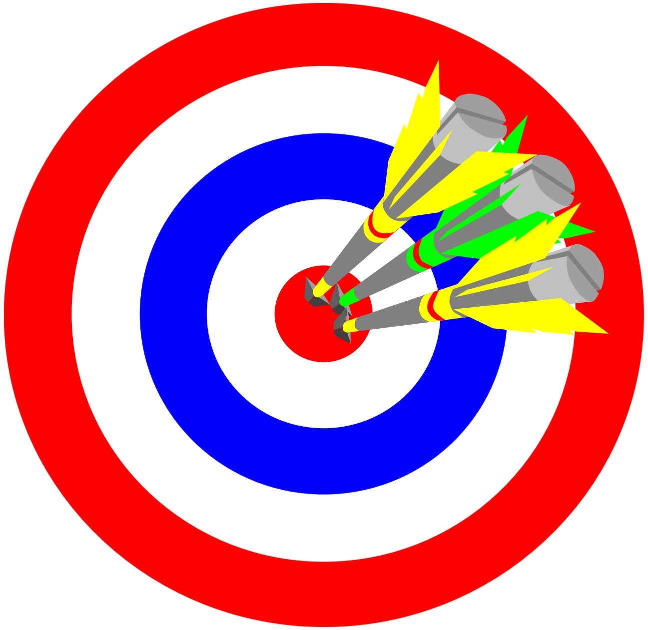
Section 14-2

Reflections

Homework Pages 580-582:

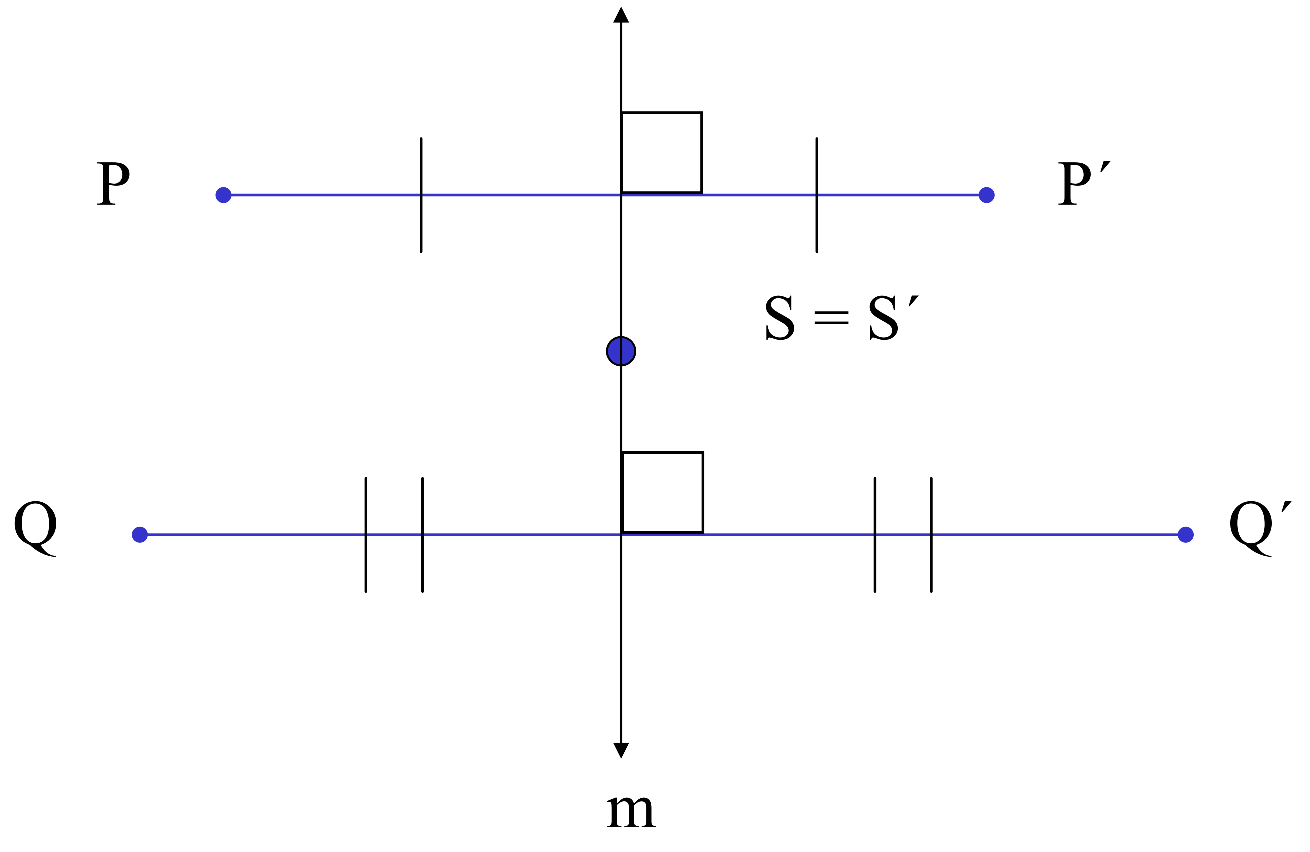
2-16 evens and 26-36 evens

Objectives

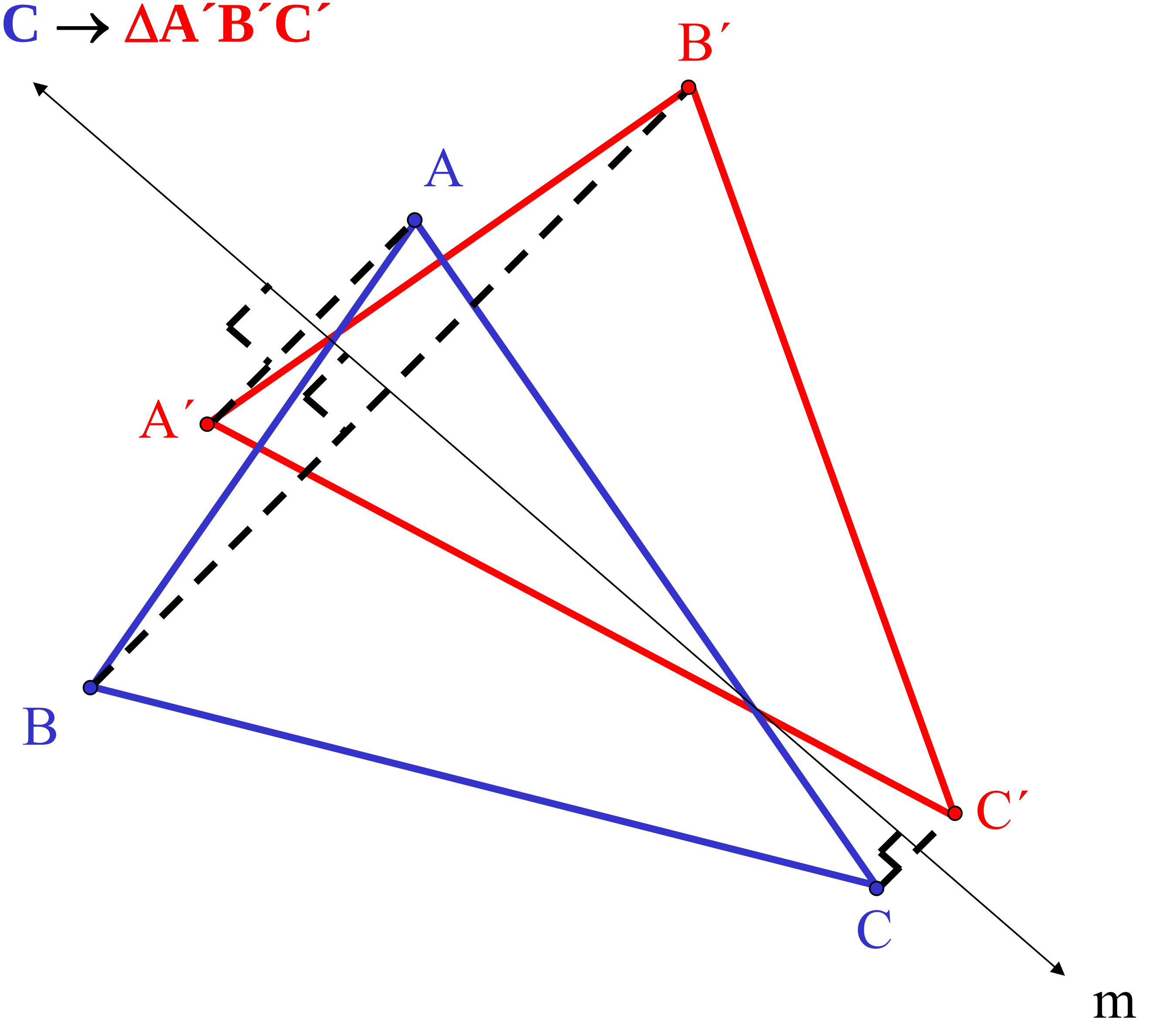


A. Understand and apply the term ‘reflection’.
B. Locate images of figures by reflection.

- ★ A reflection into line m, written R_m, maps every point P to a point P′ such that:
  - If P is not on m, then m is the perpendicular bisector of the segment connecting P to P′.
  - If P is on line m, then P = P′.
- ★ To map an object reflect its vertices.


P
P′
S = S′
Q
Q′
m

R_m: $\Delta ABC \rightarrow \Delta A'B'C'$



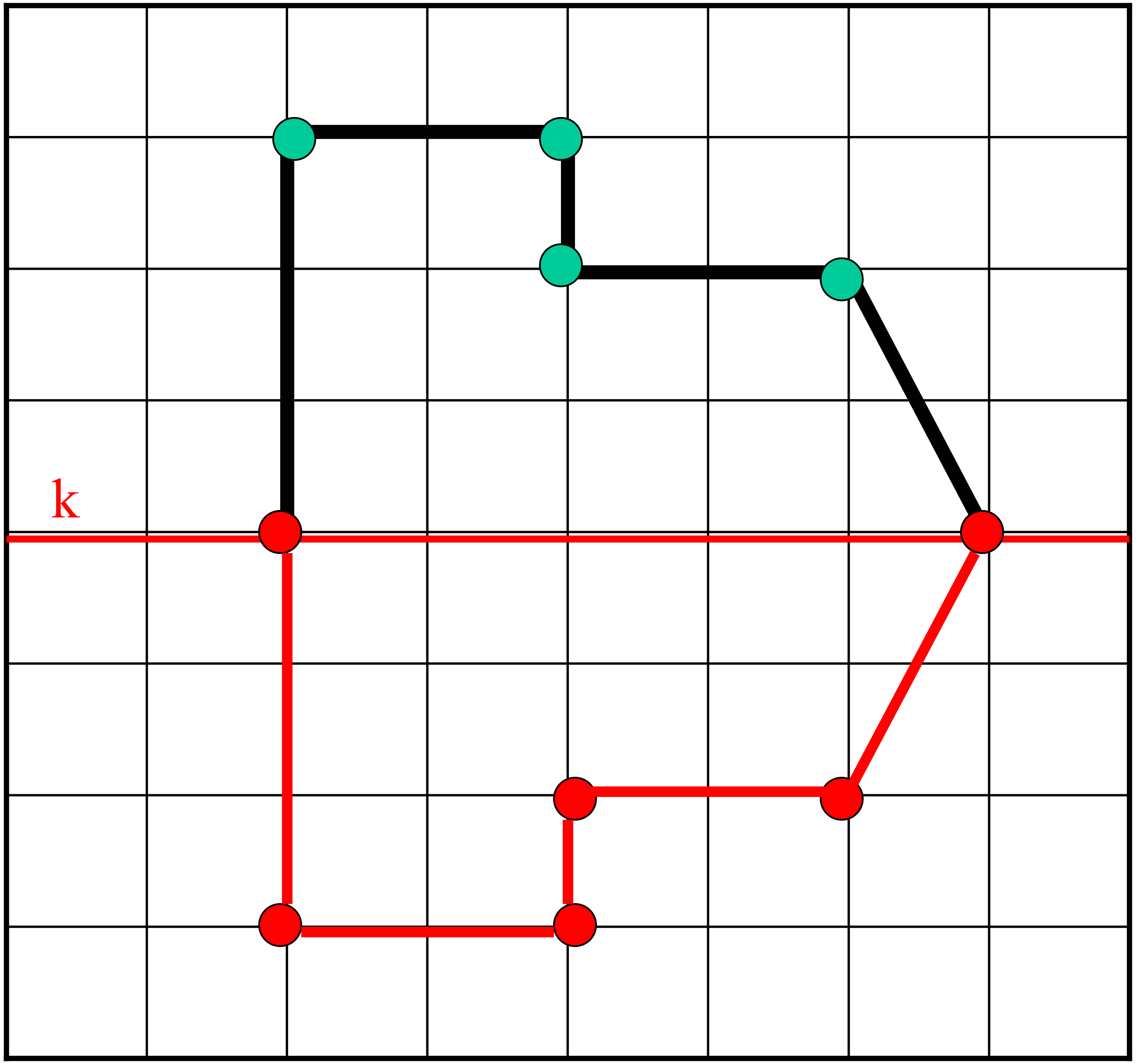
Theorem 14-2

A reflection in a line is an isometry.

R_m is a one-to-one mapping of the whole plane that preserves distance.

Sample Problems

1.



Connect points

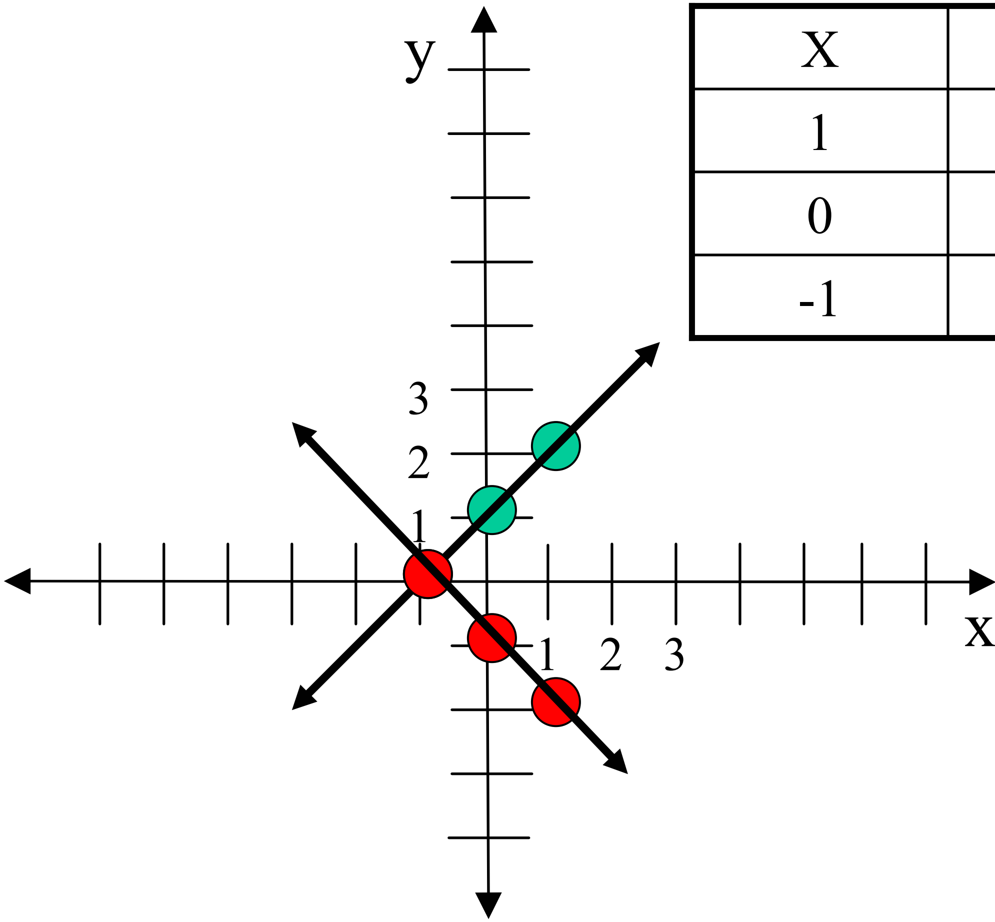
Reflect this point

Sample Problems

The line with the equation y = x + 1 is reflected in the x-axis. Find an equation of the image.

Graph the original equation (preimage).

Reflect points in x-axis and connect points.



X	Y
1	2
0	1
-1	0

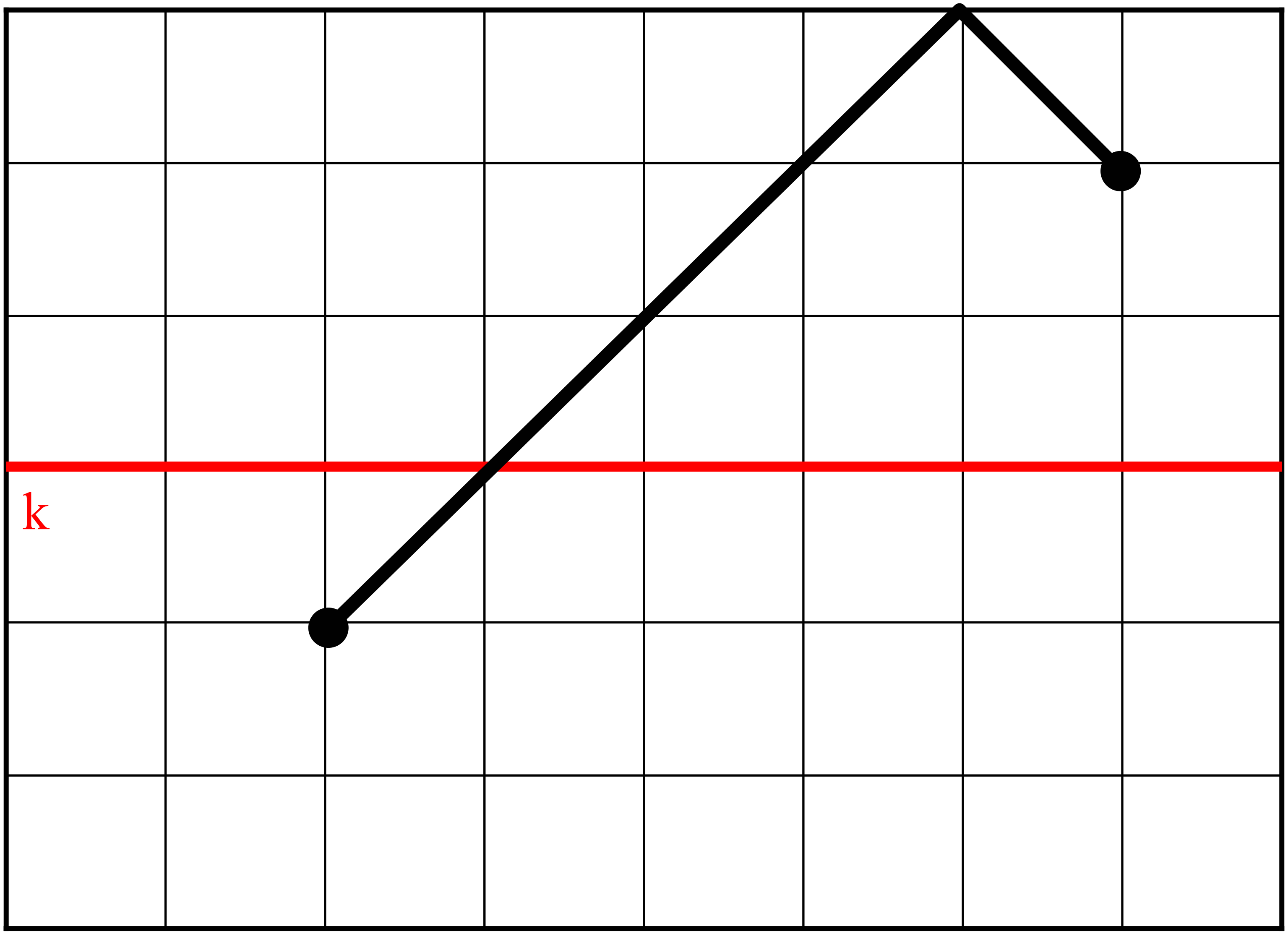
Determine equation of line.

Y-intercept (0, -1)

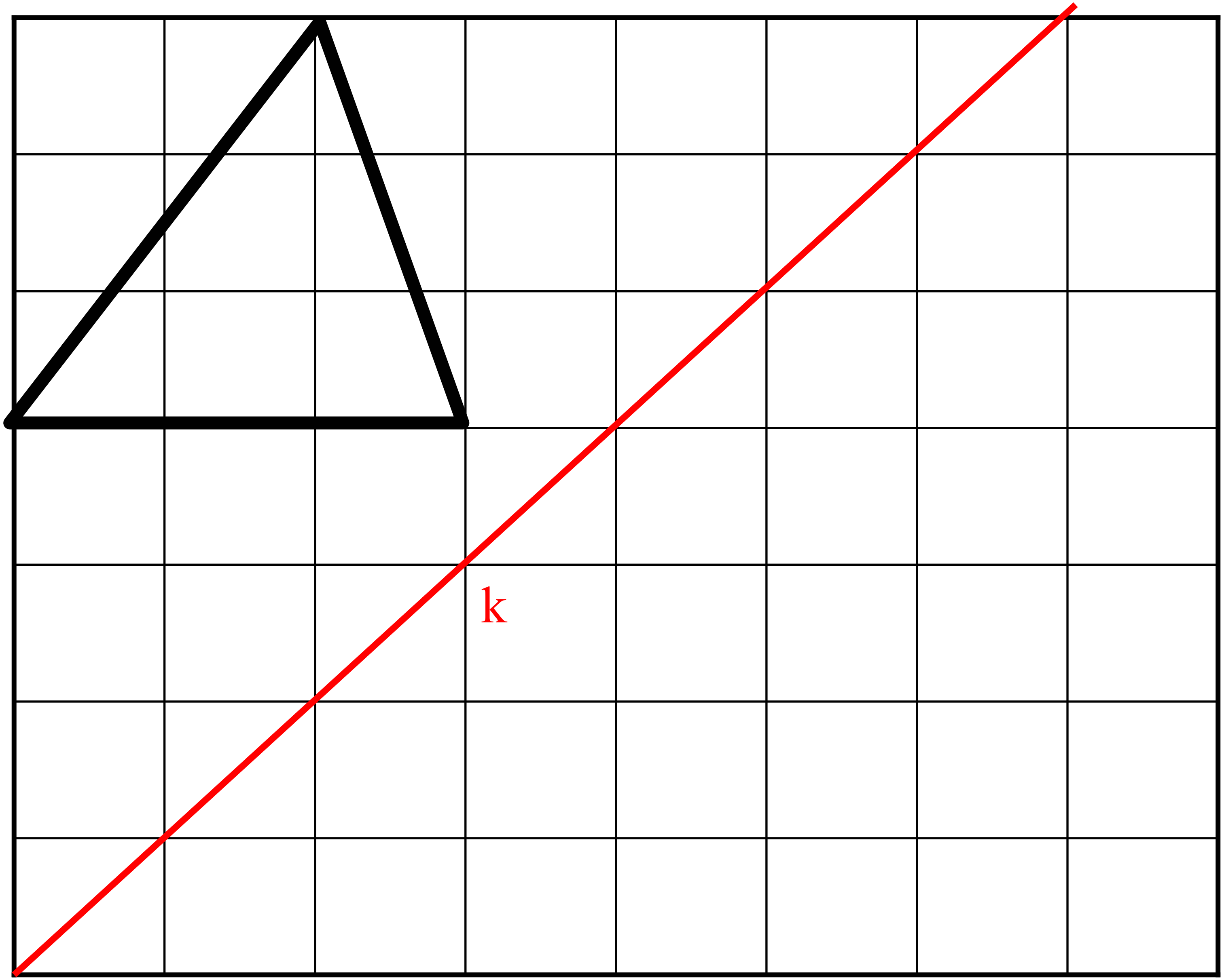
Slope = -1

Y = -x -1

Sample Problems



Sample Problems



Sample Problems

For each problem:

a) R_X

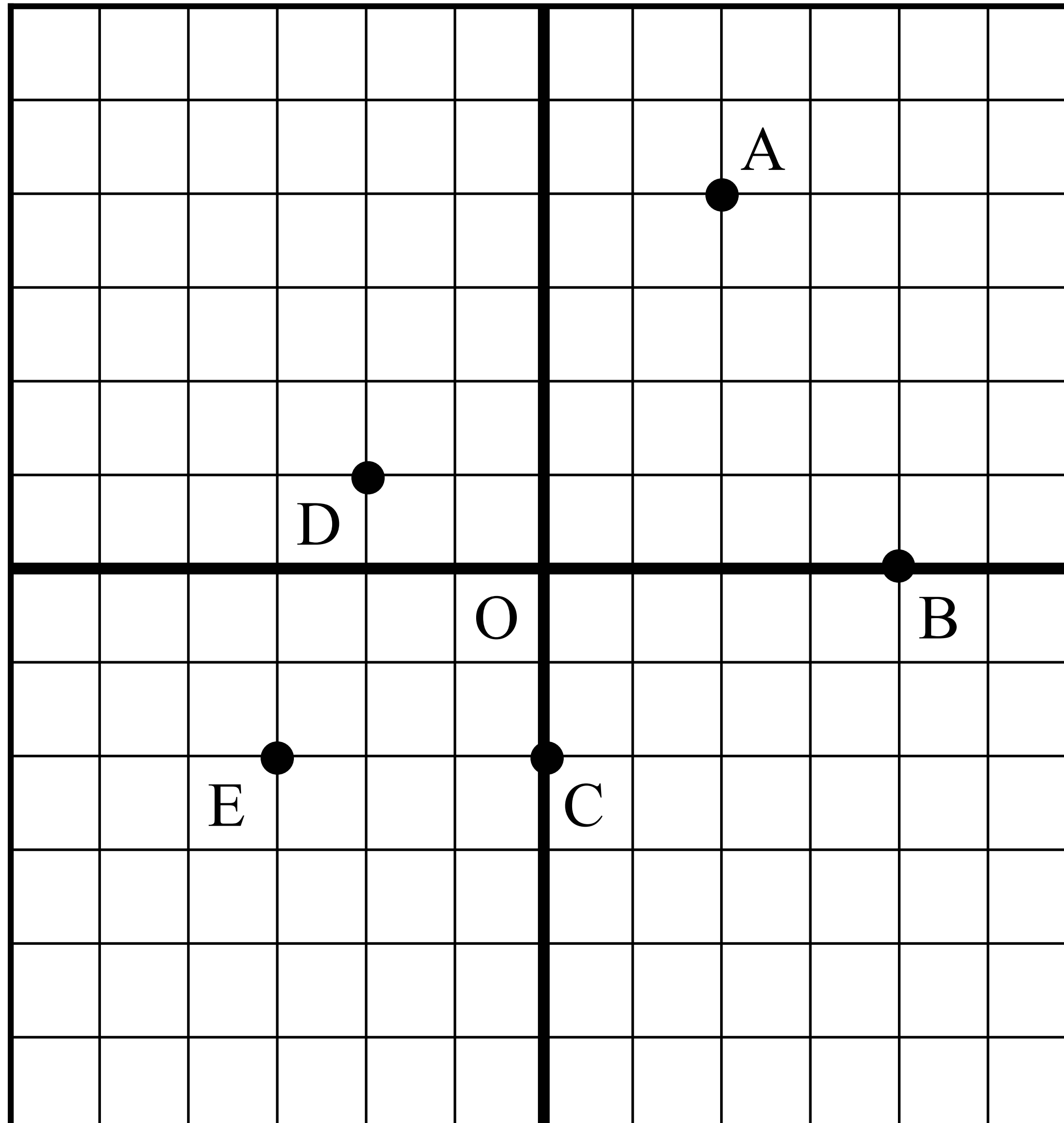
b) R_Y

c) reflection into the line y = x

7. A

9. C

11. E



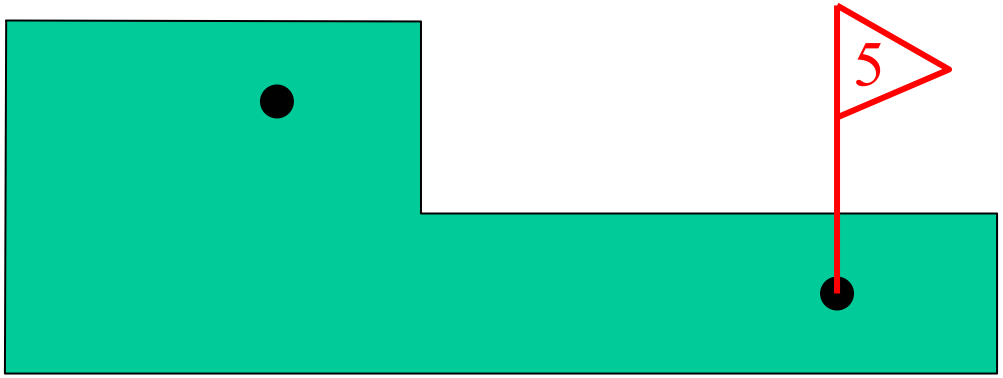
Sample Problems

13. When the word MOM is reflected into a vertical line, the image is still MOM. Find another word that can be reflected vertically and remain unchanged.

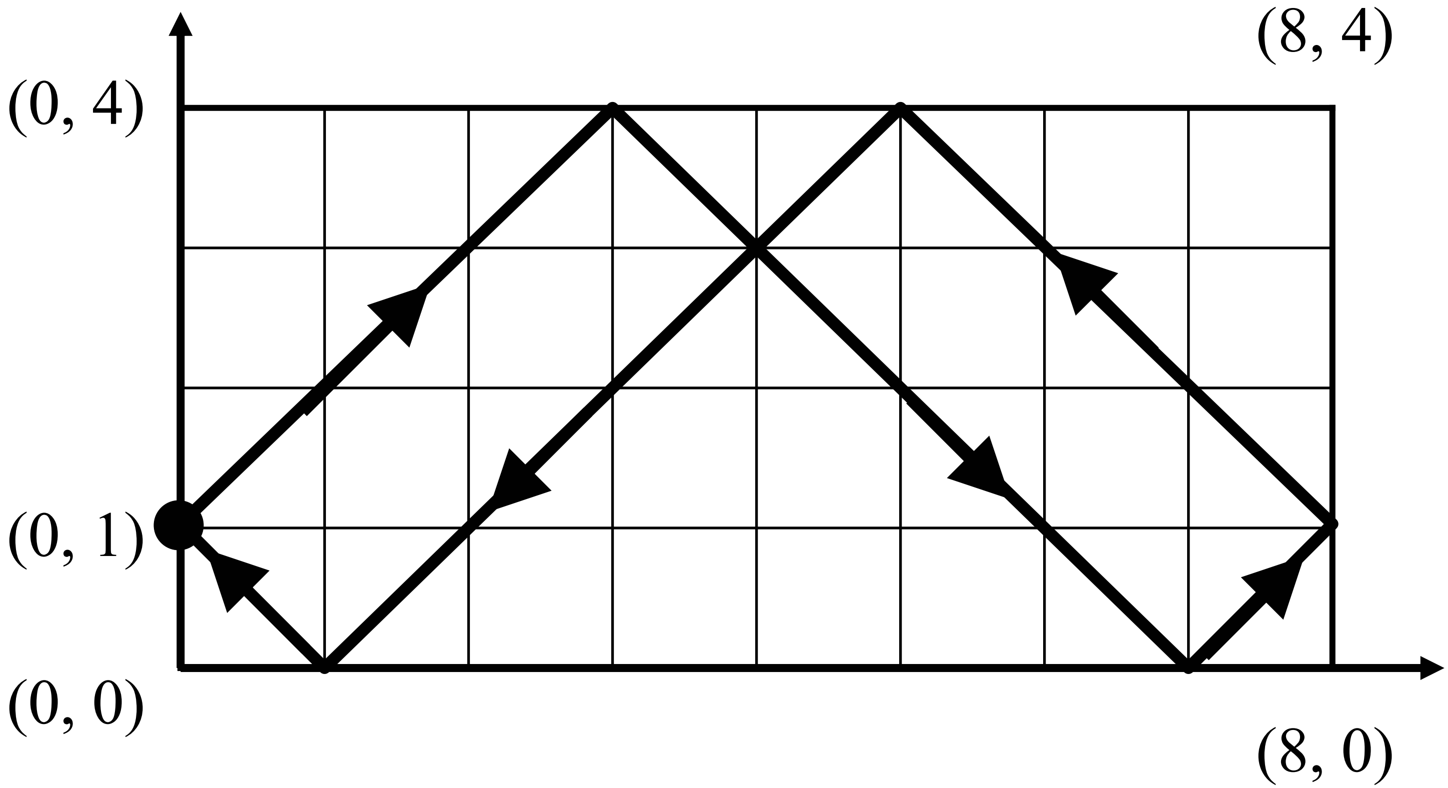
15. Draw a triangle and a line m such that R_m maps the triangle onto itself.

Sample Problems

27. Show how to score a hole in one by rolling the ball off two walls.



Sample Problems



29. A ball rolls at a 45° angle away from one side of a billiard table with a coordinate grid on it. If the ball starts at the point (0, 1) it eventually returns to this point. Would this happen if the ball started from some other point on the y-axis?

Sample Problems

31. The line with the equation $y = x + 5$ is reflected in the x-axis. Find an equation of the image.

In each exercise $R_k: A \to A'$. Find an equation of line k.

33. A(1, 4) A′(3, 4)

35. A(5, 1) A′(1, 5)

37. A(- 1, 2) A′(4, 5)

Section 14-3

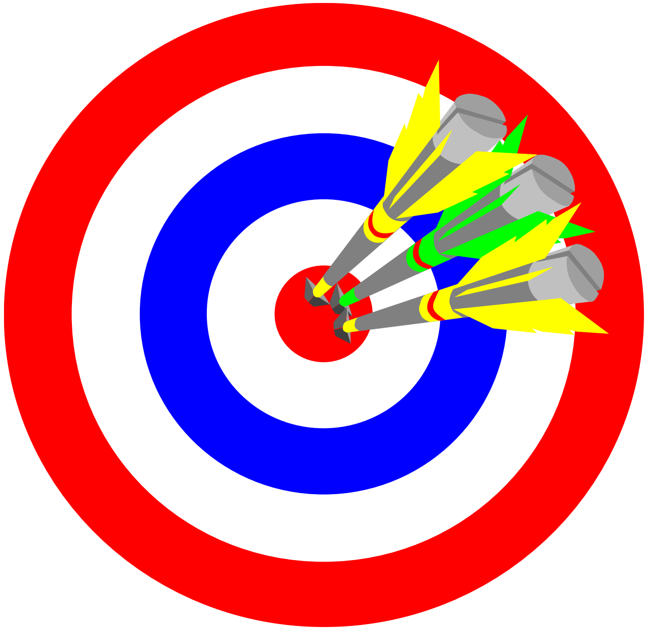
Translations and Glide Reflections

Homework Pages 586-587:

2-16 evens

Excluding 10, 12

Objectives



A. Understand and apply the terms ‘translation’ and ‘glide reflection’.

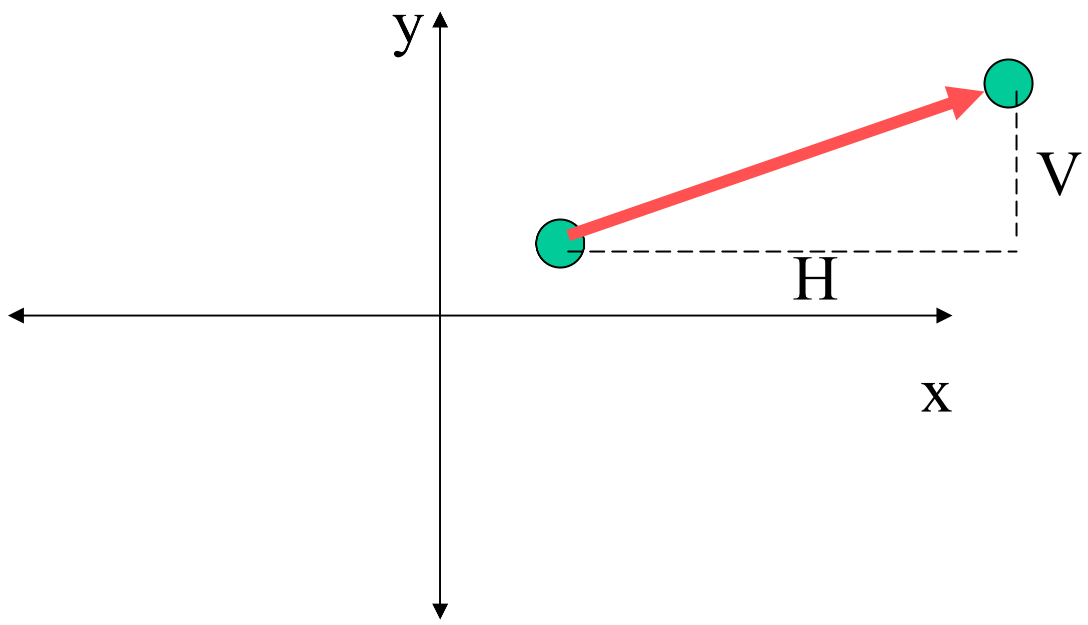
B. Locate images of figures by translation and glide reflection.

Translation (or Glide)

- Translation (or glide): a transformation that moves all points the same vector.
  - Written $T:(x, y) \rightarrow (x + H, y + V)$

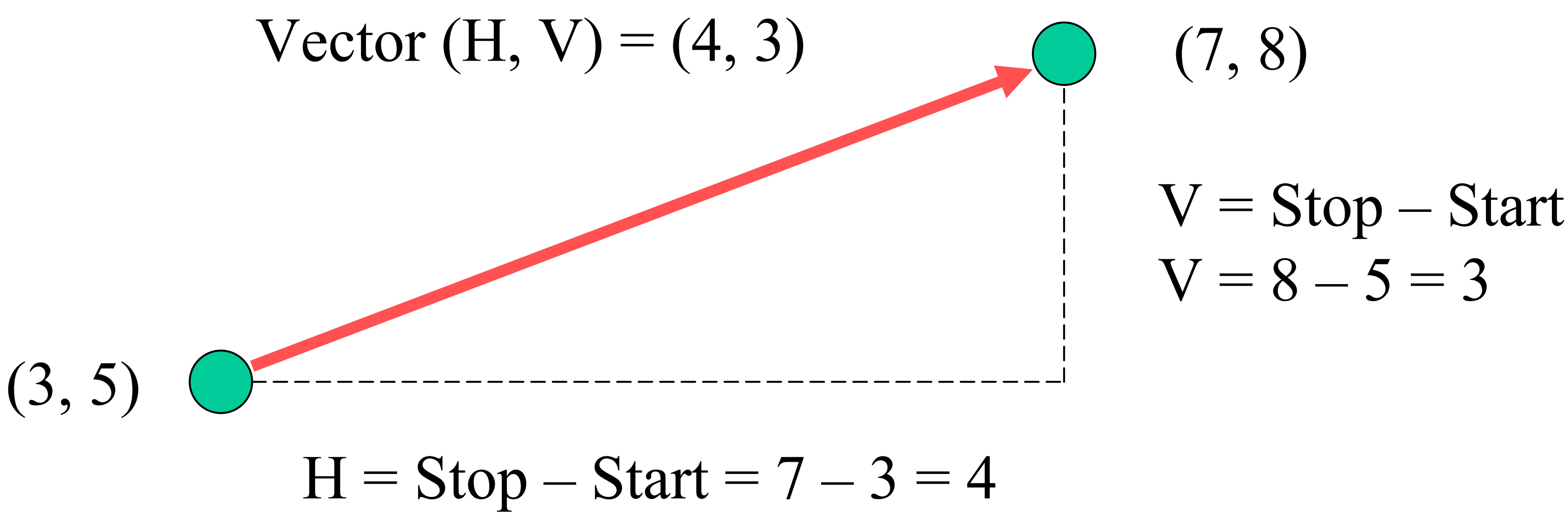
Reminders from Section 13-4: Vectors

- ★ vector: a line segment with both magnitude and direction, written as an ordered pair of numbers (H, V) where H is the horizontal component and V is the vertical component.



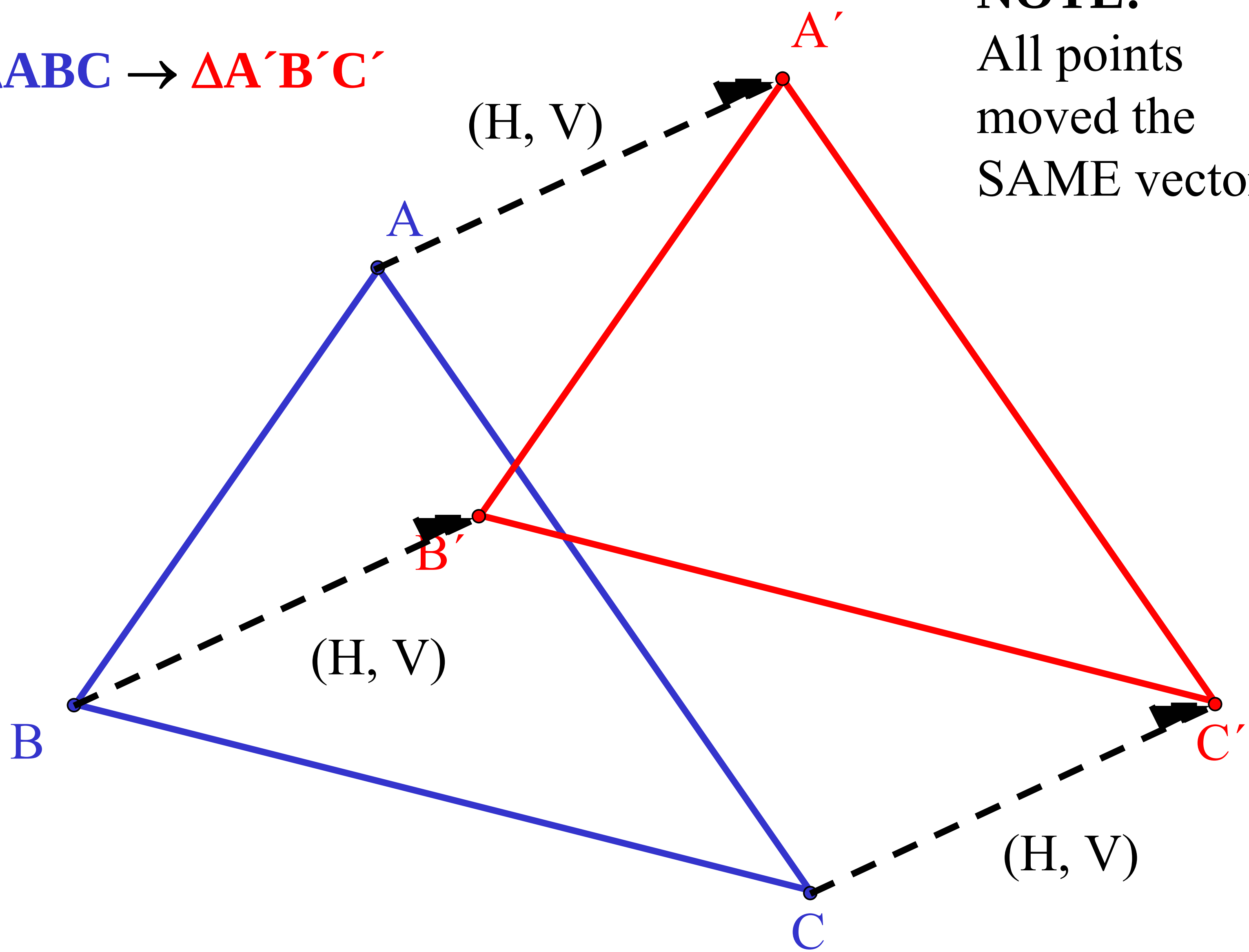
Reminders from Section 13-4: Vector Components

- horizontal component: the distance traveled left or right from the starting to the ending point, found by taking $x_{stop} - x_{start}$
- vertical component: the distance traveled up or down from the starting point to the ending point, found by taking $y_{stop} - y_{start}$



Translation Example

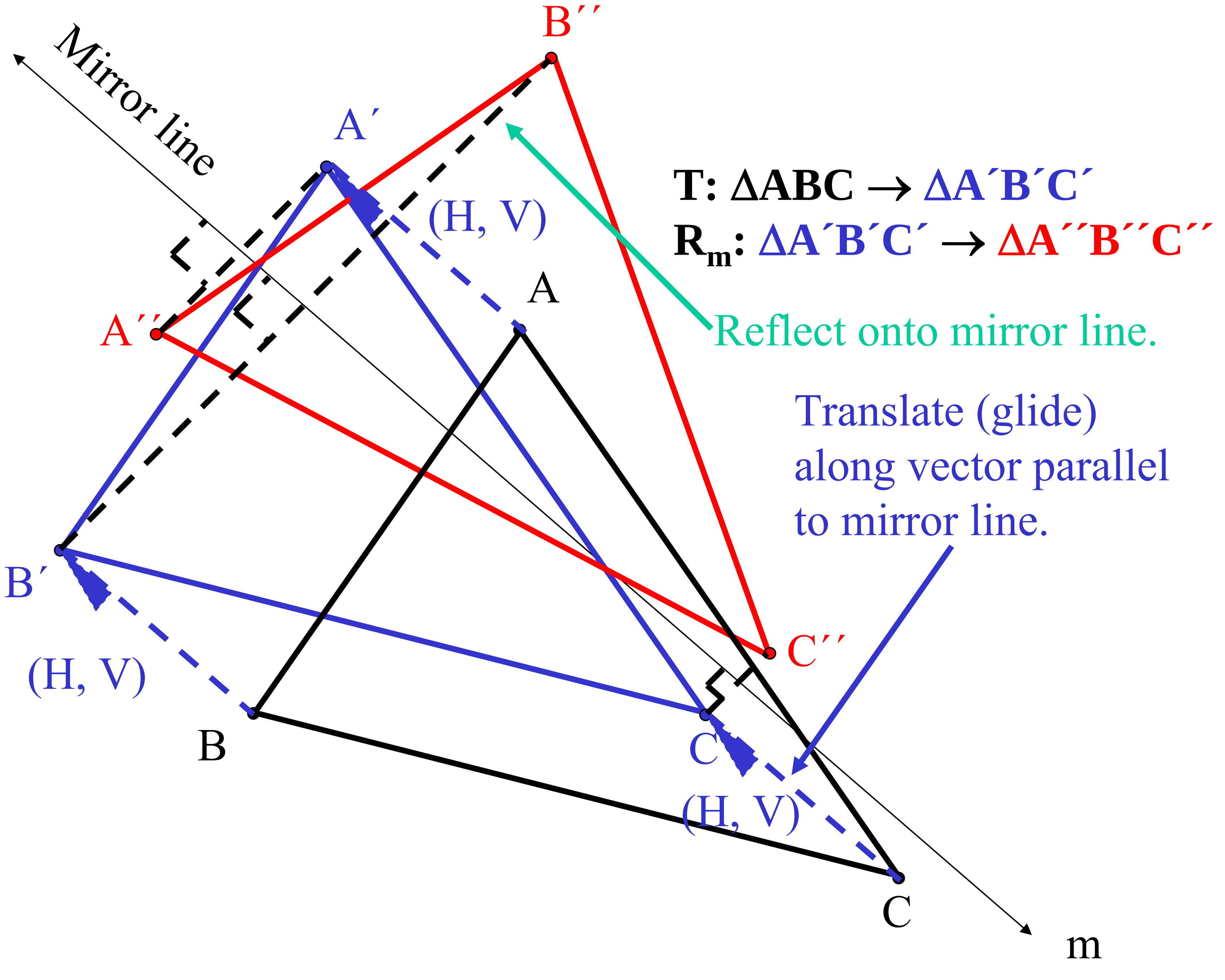
T: ΔABC → ΔA´B´C´



NOTE!
All points moved the SAME vector!

Glide Reflection

- glide reflection: a two step mapping (transformation) that first translates (glides) all points along a vector parallel to a mirror and then reflects them into the mirror.
  - 1^{st} step written $T:(x, y) \rightarrow (x + H, y + V)$
  - 2^{nd} step written R_m



Mirror line
B´´
A´
T: ΔABC → ΔA´B´C´
Rm: ΔA´B´C´ → ΔA´´B´´C´´
(H, V)
A
A´´
Reflect onto mirror line.
Translate (glide) along vector parallel to mirror line.
B´
C´´
(H, V)
B
C´
(H, V)
C
m

Theorem 14-3

A translation is an isometry.

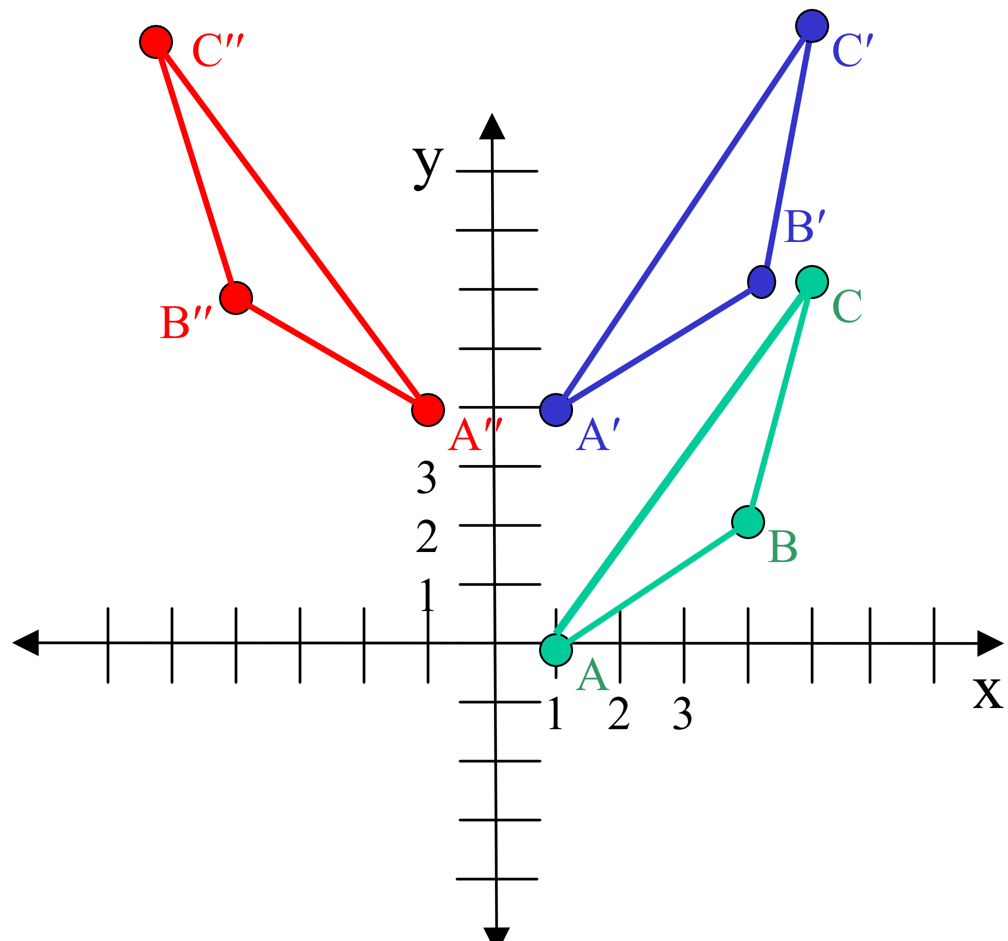
T: $(x, y) \rightarrow (x + H, y + V)$ is a one-to-one mapping, that uses the whole plane and that preserves distance.

Sample Problems

For each exercise a glide reflection is described, Graph ΔABC and its image under the glide ΔA′B′C′. Also graph ΔA′′B′′C′′ , the image of ΔA′B′C′ under the reflection.

7. Glide: All points move up 4 units. Reflection: All points are reflected into the y-axis

A(1, 0) B(4, 2) C(5, 6)



Sample Problems

For each exercise: a) Graph ΔABC and its image ΔA′B′C′ and answer the question is ΔABC ≅ ΔA′B′C′? b) Draw arrows from A to A′, B to B′, and C to C′. c) Are your arrows the same length? Are they parallel?

1. T: $(x, y) \rightarrow (x - 2, y + 6)$ A(- 2, 0) B(0, 4) C(3, - 1)

3. If T: $(0, 0) \rightarrow (5, 1)$ then T: $(3, 3) \rightarrow (?, ?)$
5. If T: $(- 2, 3) \rightarrow (2, 6)$ then T: $(?, ?) \rightarrow (0, 0)$

Sample Problems

9. Where does the glide reflection in exercise 7 map (x, y)

11. Which of the following properties are invariant under a translation? a) distance b) angle measure c) area d) orientation

13. Translations R and S are described. R maps a point P to P′, and S maps P′ to P′′. Find T, the translation that maps P to P′′. R: $(x, y) \rightarrow (x + 1, y + 2)$

    S: $(x, y) \rightarrow (x - 5, y + 7)$

    T: $(x, y) \rightarrow (?, ?)$

15. If a translation T maps P to P′, then T can be described by the vector PP′. Suppose a translation T is described by the vector (3, - 4) because it glides all points 3 units right and 4 units down. a) Graph points A(- 1, 2) B(0, 6), A′ and B′. b) What kind of figure is AA′B′B? What is its perimeter?

Section 14-4

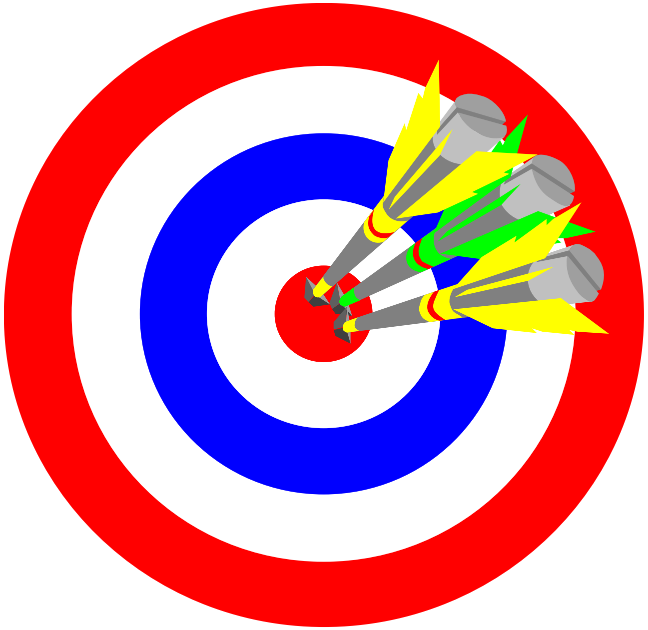
Rotations

Homework Pages 590-591:

2-34 evens

Excluding 24, 26

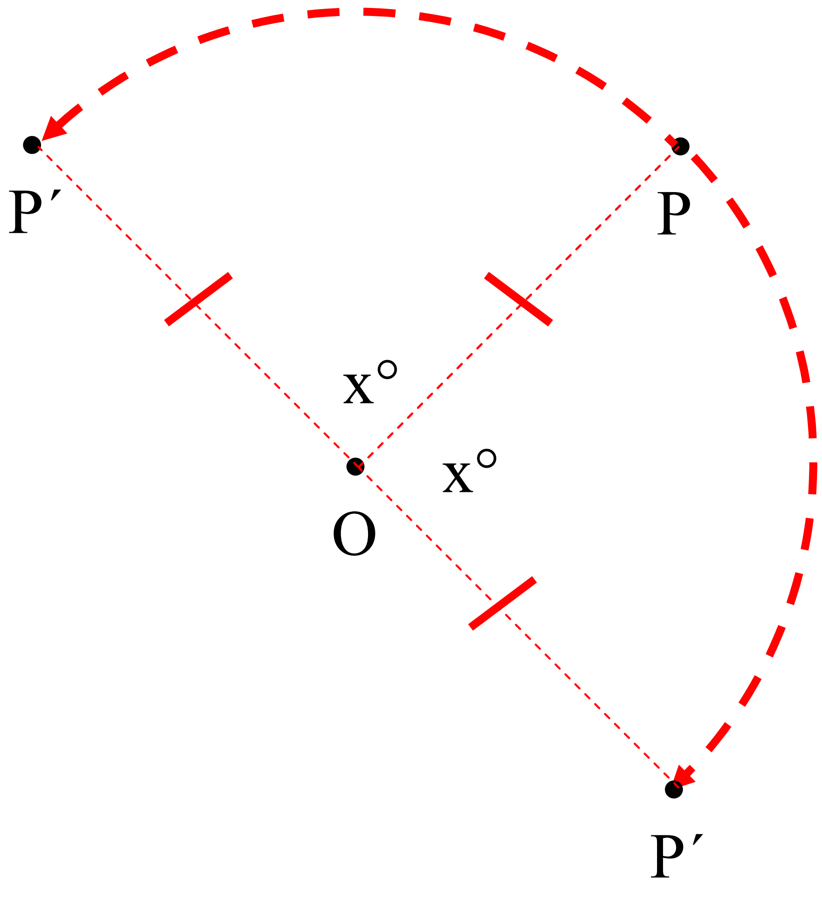
Objectives

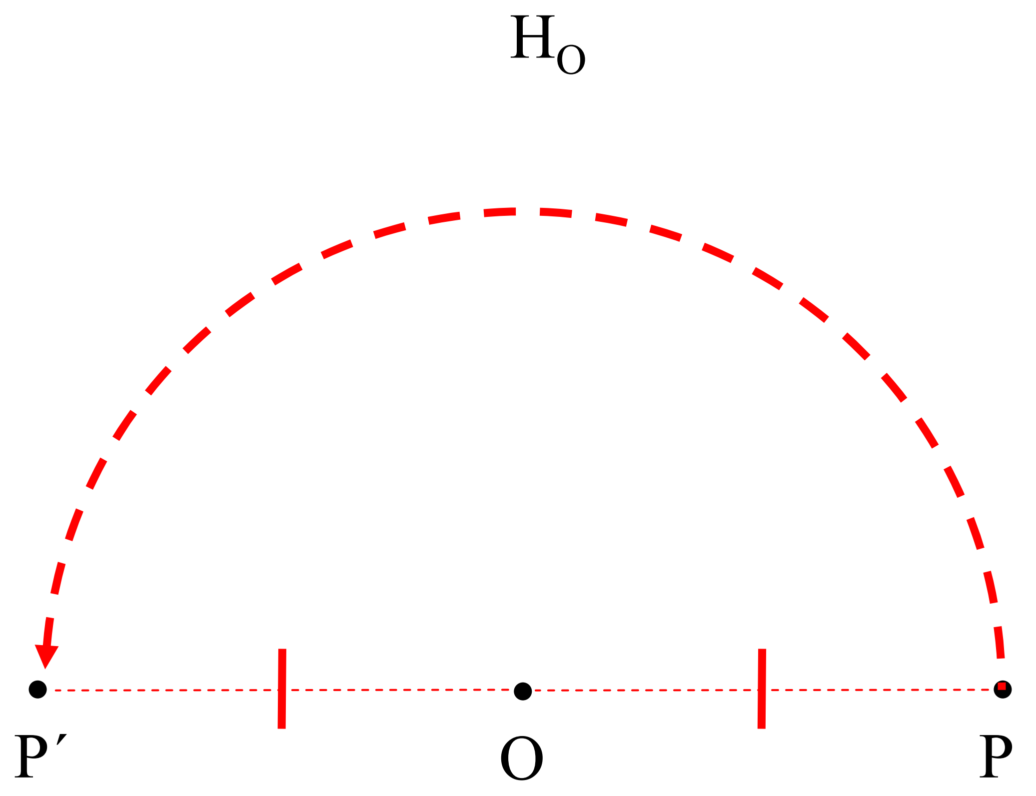


A. Understand and apply the term ‘rotation’.
B. Locate images of figures by rotation.

- ★ rotation: a mapping, written $\Re_{O, x}$, which moves points around a stationary point O, x degrees such that:
  - If point P is different from O, then OP = OP′.
  - If point P is the point O then P = P′.
  - If x > 0 then the rotation moves counter-clockwise.
  - If x < 0 then the rotation moves clockwise.
- ★ half-turn: written H_O, is a rotation of 180 degrees around the point O.
  - a half-turn clockwise = a half-turn counter-clockwise.
- ★ During homework and tests, make sure to CLEARLY distinguish between:
  - $\Re$ which indicates rotation, and
  - R which indicates reflection.

$\Re_{O,\,x}$, where $x > 0$. $\Re_{O,\,x}$, where $x < 0$.





H_O
P′
O
P

Theorem 14-4

A rotation is an isometry.

$\Re_{O,\, x}$ is a one-to-one mapping of the whole plane that preserves distance.

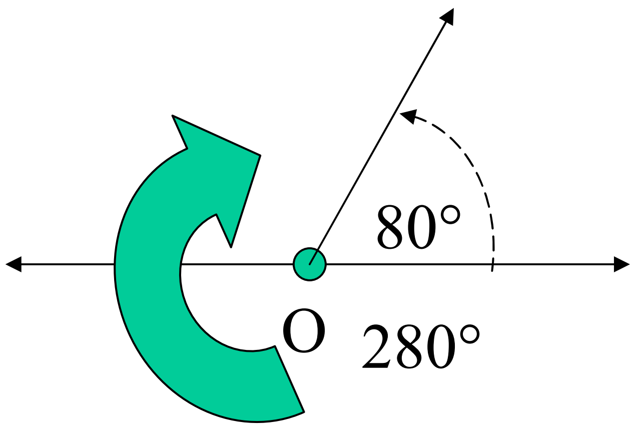
Sample Problems

State another name for each rotation.

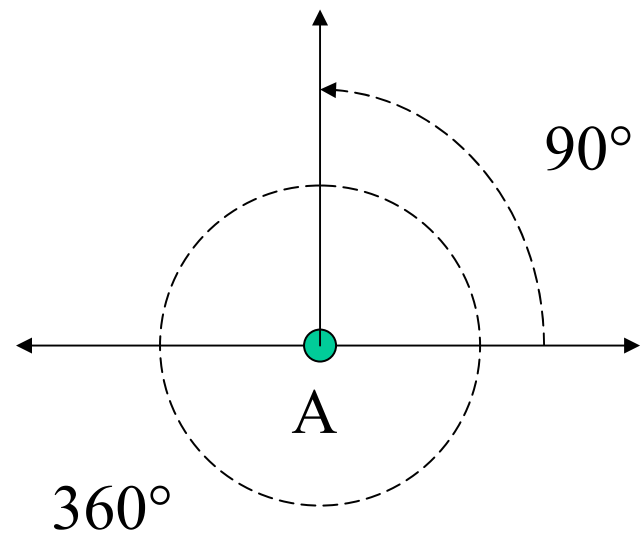
1. $\Re_{O,80}$

3. $\Re_{A,\,450}$

5. H_O



$\Re_{O,-280}$



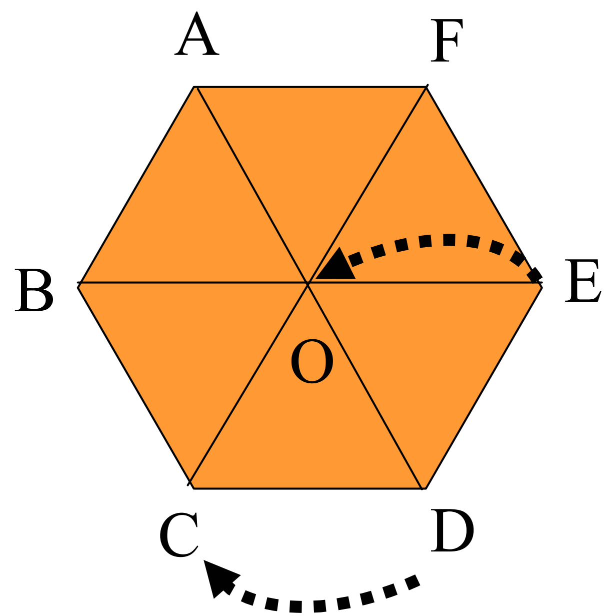
$\Re_{A,90}$

Sample Problems

The diagonals of regular hexagon ABCDEF form six equilateral triangles as shown. Complete each statement below

7. $\mathfrak{R}_{O,-60}$: $D \rightarrow ?$

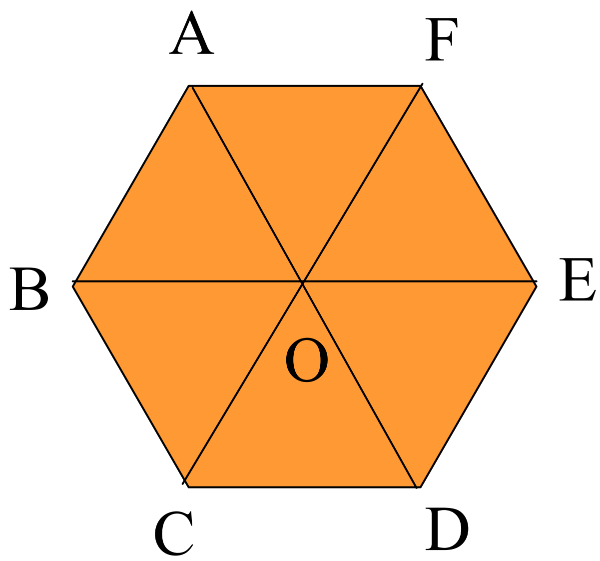
9. $\mathfrak{R}_{D,60}$: $? \rightarrow O$



Sample Problems

The diagonals of regular hexagon ABCDEF form six equilateral triangles as shown. Complete each statement below

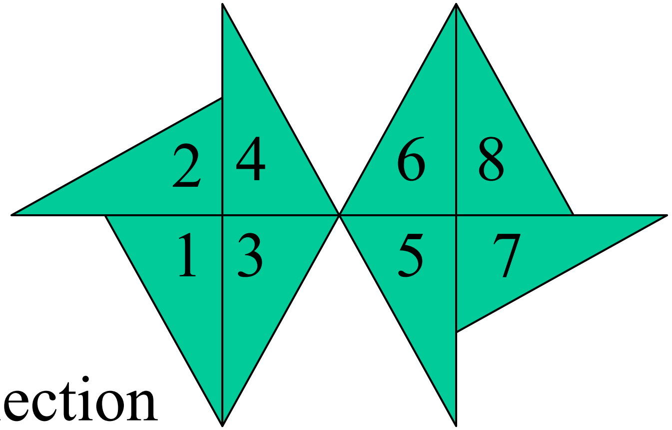
11. $H_O(A) = ?$
13. If k is the perpendicular bisector of FE, then $R_k(A) = ?$



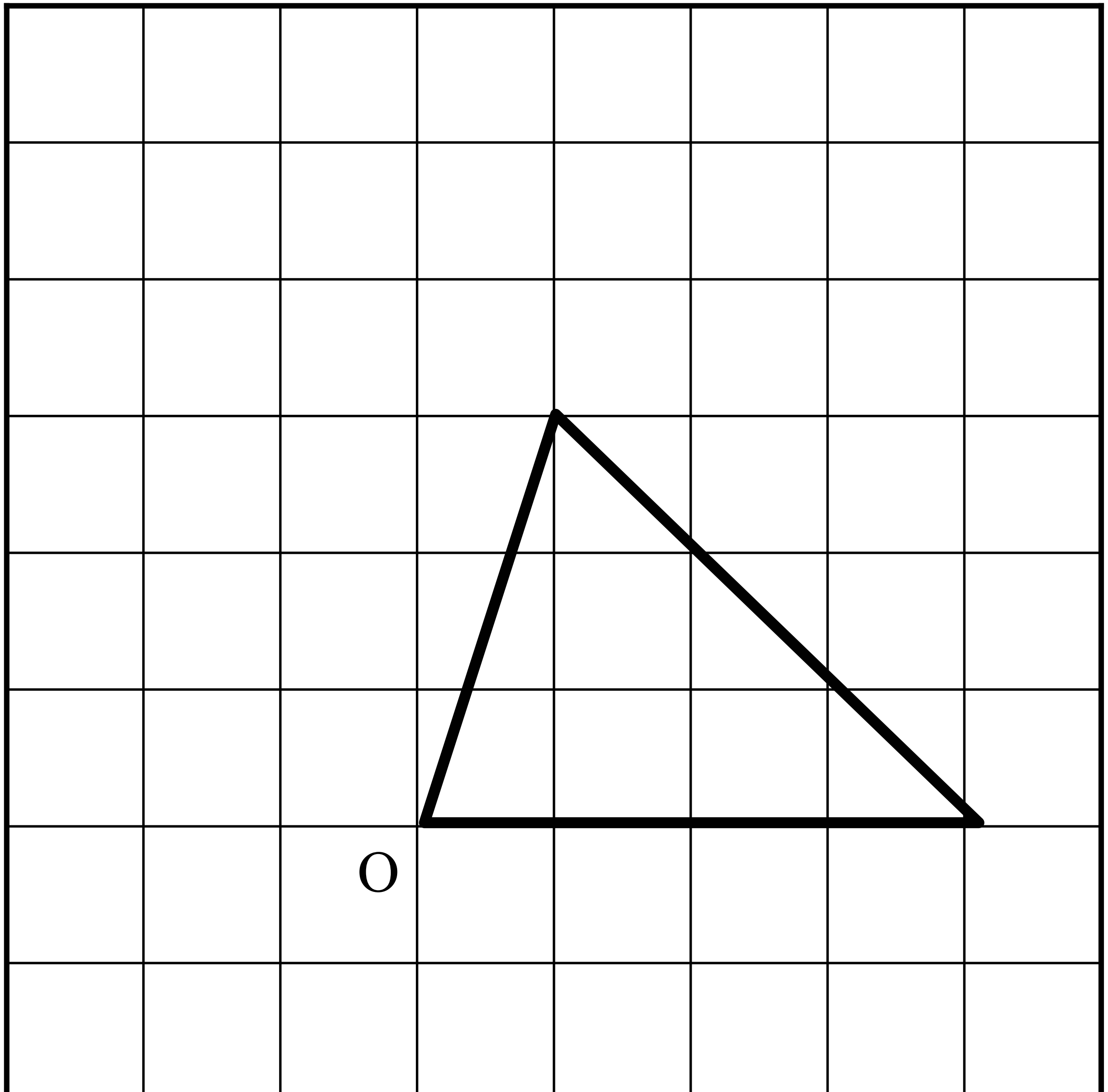
Sample Problems

State whether the specified triangle is mapped to the other triangle by a reflection, translation, rotation or half-turn.

15. 1 to 2
17. 1 to 4
19. 2 to 4
21. 4 to 6
23. Is there a glide reflection that maps triangle 1 to triangle?
25. Which of the following properties are invariant under a half turn? a) distance b) angle measure c) area d) orientation

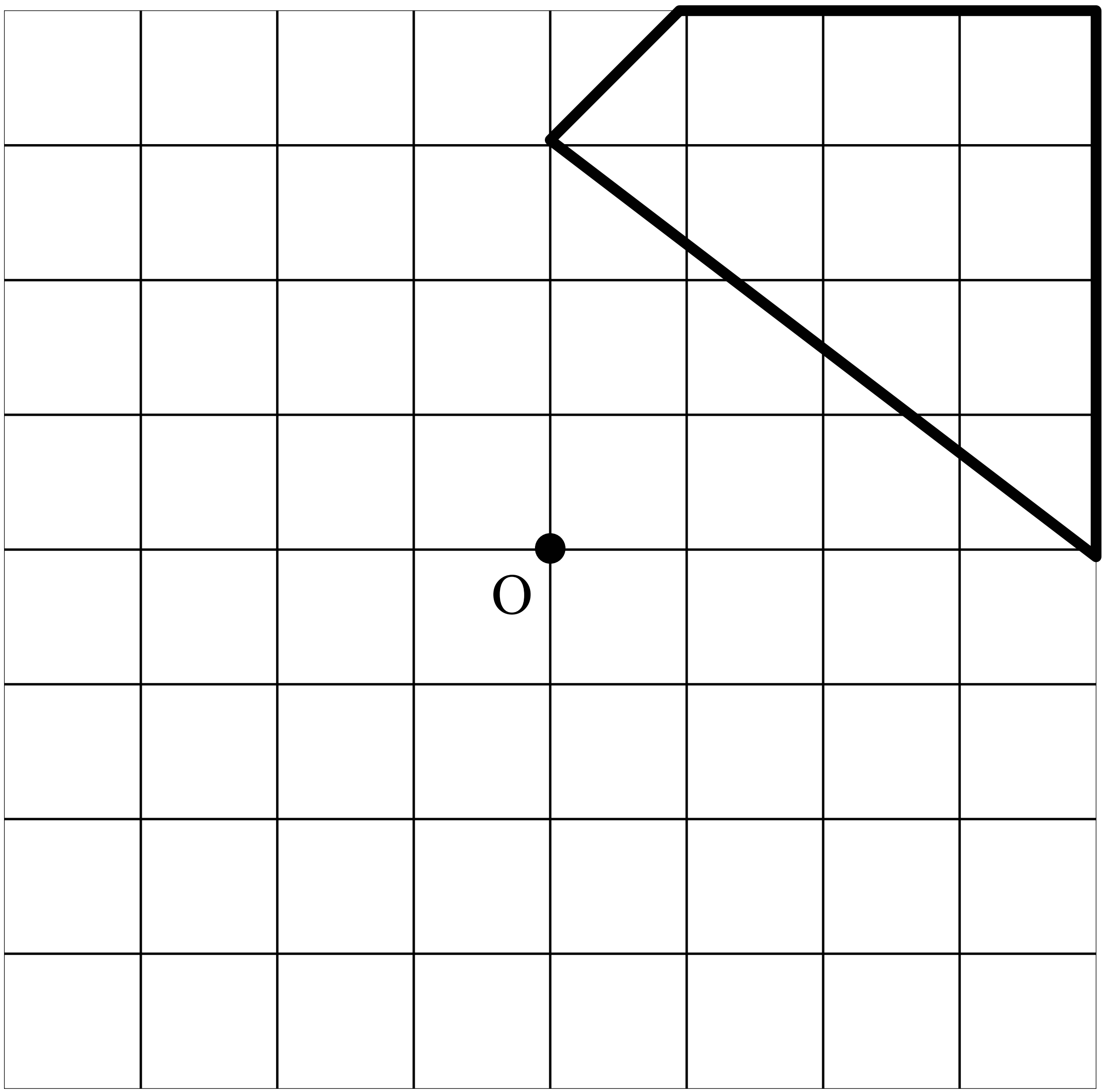


Sample Problems



27. $\Re_{O,\ 90}$

Sample Problems



29. H_O

Section 14-5

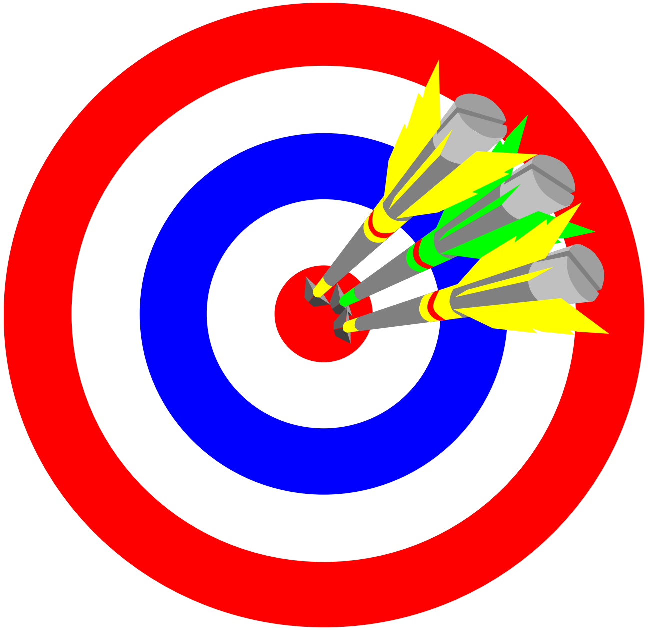
Dilations

Homework Pages 596-597:

2-24 evens

Excluding 16

Objectives



A. Understand and apply the term ‘dilation’.
B. Understand and apply the terms ‘expansion’ and ‘contraction’ in reference to dilations.
C. Locate images of figures by dilation.

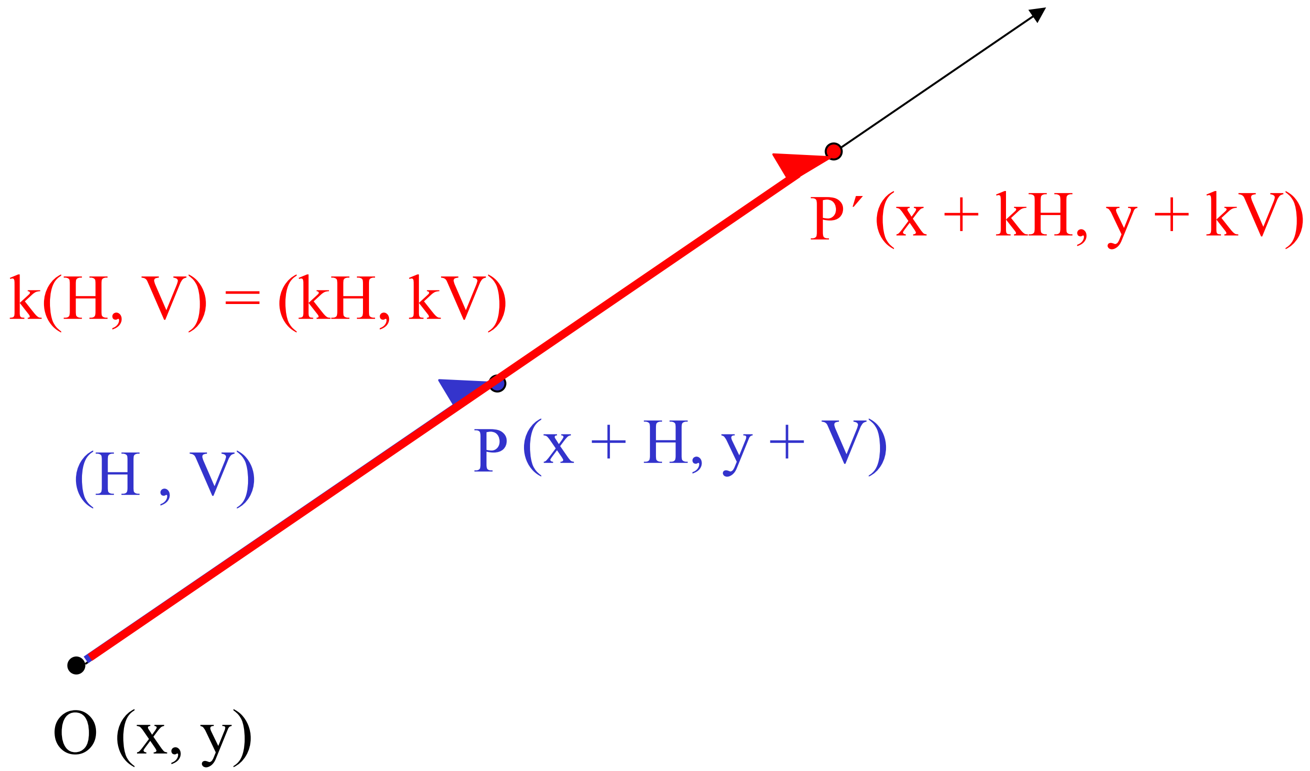
The Progression

- When we discussed triangles
  - we first discussed congruence of triangles
  - we then discussed SIMILARITY of triangles
- During our discussion of mappings
  - we first discussed congruence mappings
    - Reflection
    - Translation
    - Rotation
  - we now discuss SIMILARITY mappings
    - Dilations

- ★ dilation: the dilation $D_{O,k}$ is a similarity mapping with center at point O and with a scale factor of k that maps any point P to its image P′ such that:
  - If $k > 0$, P′ lies on the ray OP and $OP' = k(OP)$
  - If $k < 0$, then P′ lies on the ray opposite to OP and $OP' = |k|(OP)$
  - The center O is its own image.
- ★ expansion: a dilation where $|k| > 1$.
- ★ contraction: a dilation where $|k| < 1$.
- ★ similarity mapping: a mapping where the preimage and image are similar.

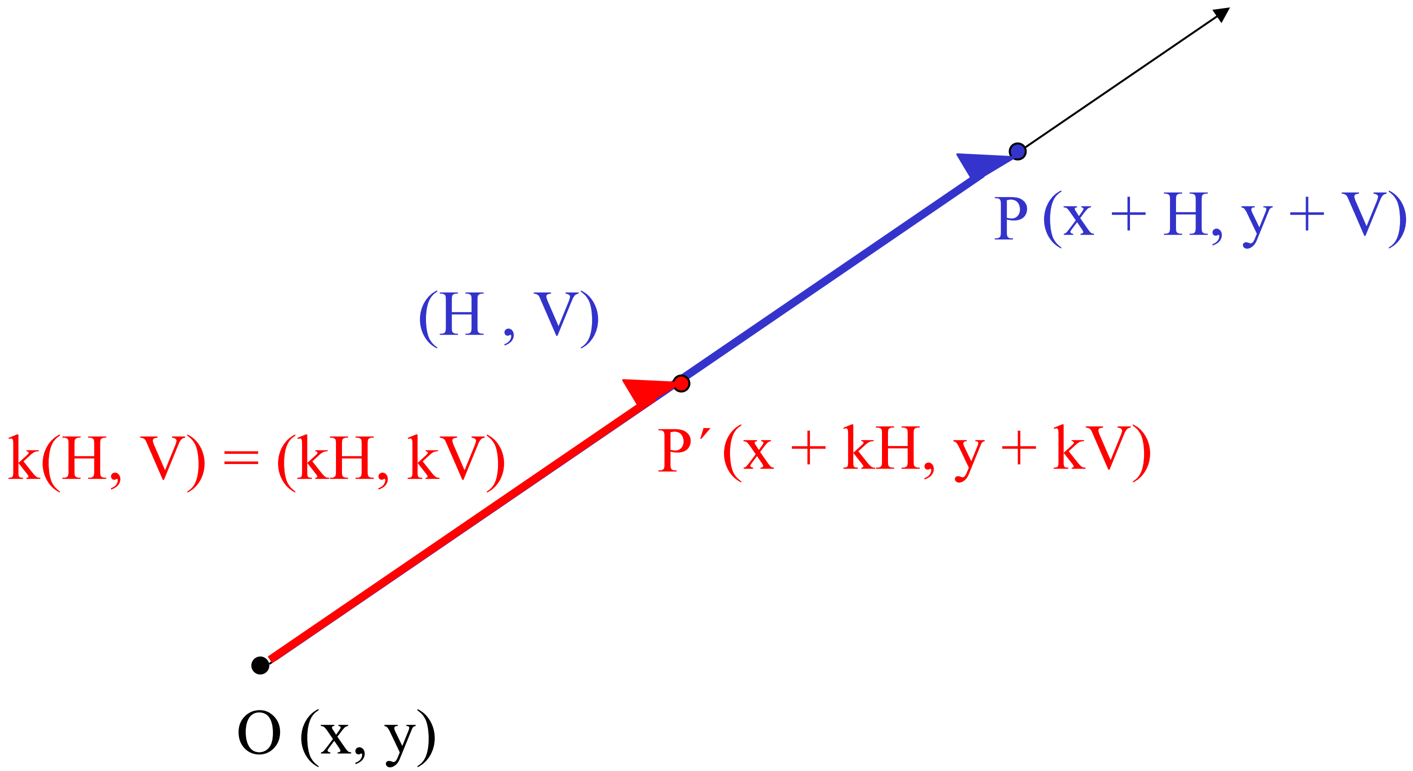
Same-side Expansion

$D_{O,k}$ where $k > 0$ & $|k| > 1$



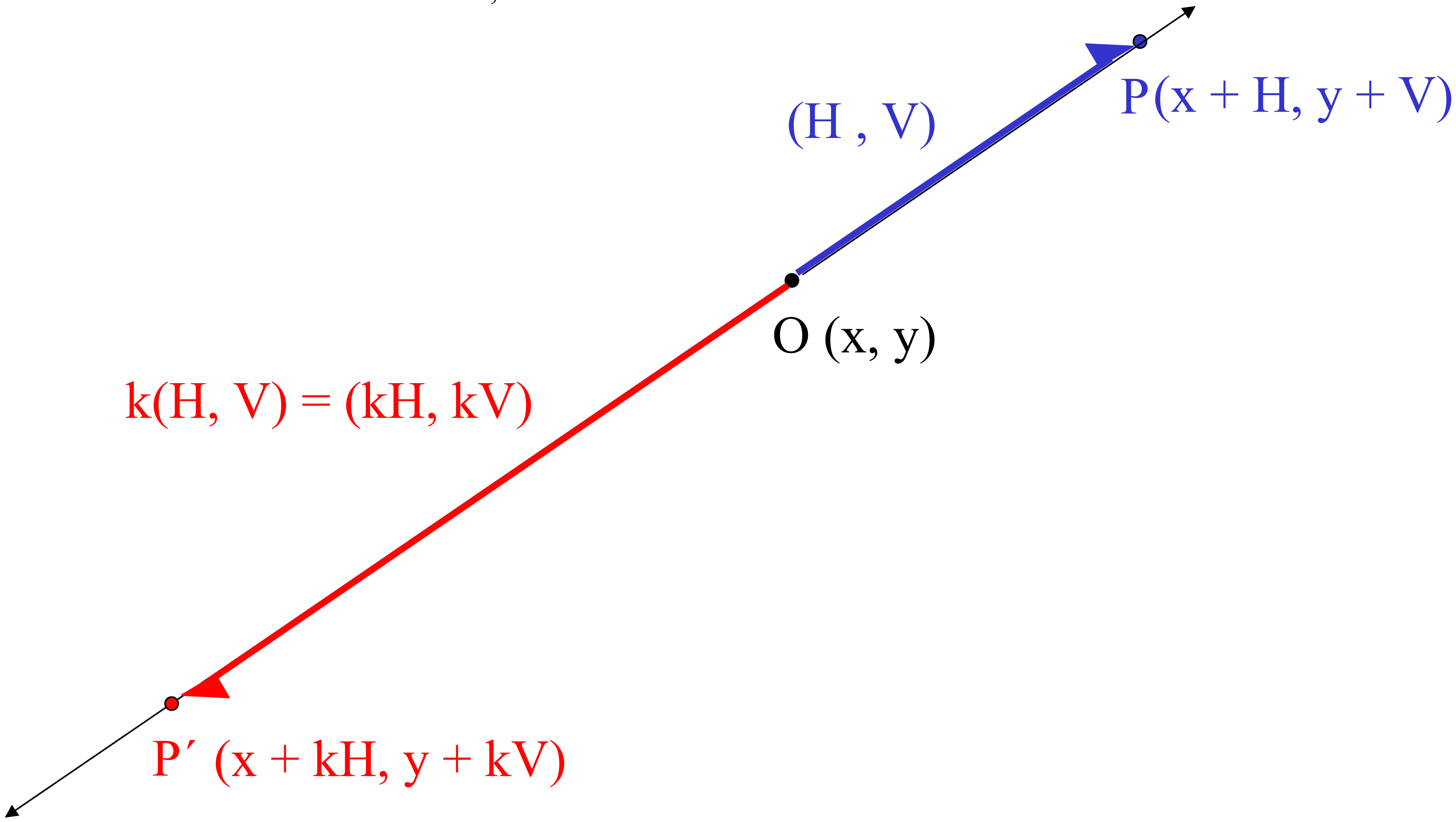
Same-side Contraction

$D_{O,\,k}$ where $k > 0$ & $|k| < 1$



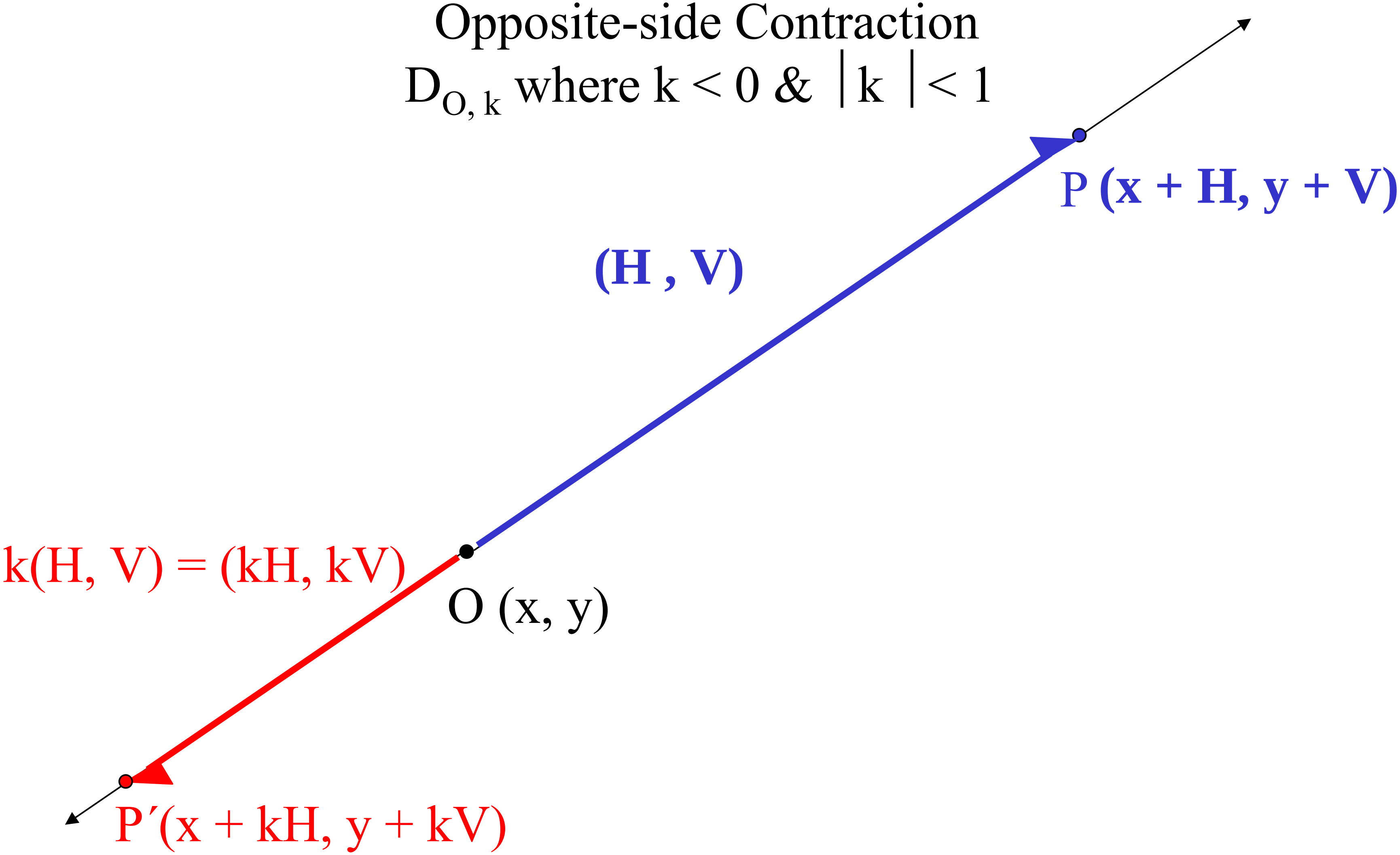
Opposite-side Expansion

$D_{O,\,k}$ where $k < 0$ & $|k| > 1$



Opposite-side Contraction

$D_{O,\,k}$ where $k < 0$ & $|k| < 1$



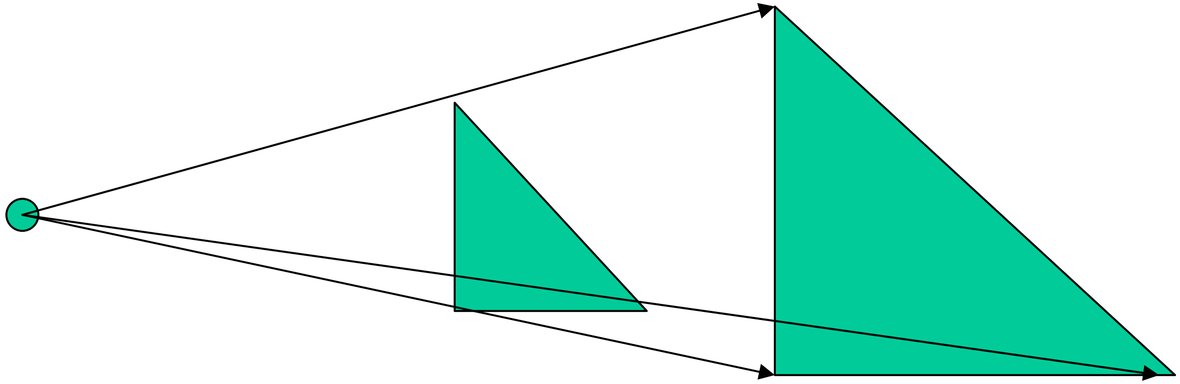
Steps for Dilation

1. Locate the center of the dilation.
2. Determine the direction of the dilation.
   a) If $k > 0$, then the dilation is in the same direction as the direction from the center to each particular point.
   b) If $k < 0$, then the dilation is in the opposite direction as the direction from the center to each particular point.
3. Determine if the dilation is a contraction, congruence, or expansion.
   a) If $|k| < 1$, then the dilation is a contraction.
   b) If $|k| = 1$, then the dilation is a congruence.
   c) If $|k| > 1$, then the dilation is an expansion.

Theorem 14-5

A dilation maps any triangle to a similar triangle.

$D_{O,k}: \Delta ABC \rightarrow \Delta A'B'C'$ then $\Delta ABC \sim \Delta A'B'C'$



Theorem 14-5 Corollary 1

A dilation maps an angle to a congruent angle.

$D_{O,k}: \angle A \rightarrow \angle A'$ then $\angle A \cong \angle A'$

Theorem 14-5 Corollary 2

A dilation $D_{O,\,k}$ maps any segment to a parallel segment $|k|$ times as long.

If $D_{O,k}$: $AB \rightarrow A'B'$ then $AB \parallel A'B'$ and $A'B' = |k|\,(AB)$

Theorem 14-5 Corollary 3

A dilation $D_{O, K}$ maps any polygon to a similar polygon whose area is k^2 times as large.

If $D_{O,k}$: POLYGON → P′O′L′Y′G′O′ N′ then
POLYGON ~ P′O′L′Y′G′O′ N′ and
Area of P′O′L′Y′G′O′ N′ = (k^2)(Area of POLYGON)

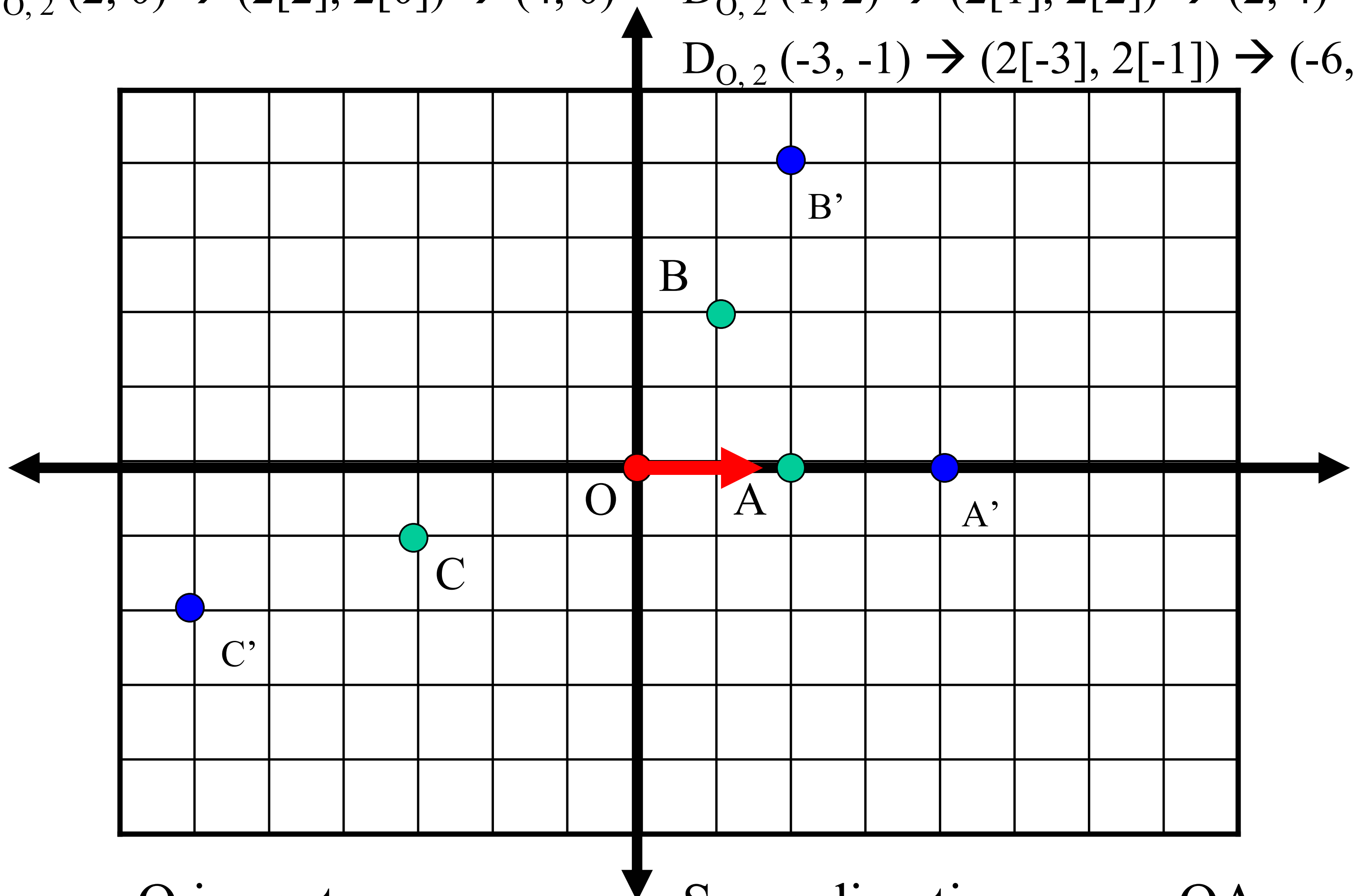
Sample Problems

1. $D_{O,\,2}$ k = 2, therefore |k| > 1, therefore expansion.

$D_{O,\,2}$ (2, 0) → (2[2], 2[0]) → (4, 0)

$D_{O,\,2}$ (1, 2) → (2[1], 2[2]) → (2, 4)

$D_{O,\,2}$ (-3, -1) → (2[-3], 2[-1]) → (-6, -2)



O is center.

Same direction as ray OA.

Sample Problems

A dilation with the origin as its center maps the given point to the given image. Find the scale factor of the dilation. Is the dilation an expansion or a contraction?

9. $(2, 0) \rightarrow (8, 0)$

Rewrite as $D_{O,\,k}\,(2, 0) \rightarrow (8, 0)$

$D_{O,\,k}\,(2, 0) \rightarrow ([k]2, [k]0)$

$[k]2 = 8, [k]0 = 0$

$k = \text{scale factor} = 4$

$D_{O,\,4}\,(2, 0) \rightarrow (8, 0)$

$|k| > 1$, expansion

11. $(3, 9) \rightarrow (1, 3)$

Rewrite as $D_{O,\,k}\,(3, 9) \rightarrow (1, 3)$

$D_{O,\,k}\,(3, 9) \rightarrow ([k]3, [k]9)$

$[k]3 = 1, [k]9 = 3$

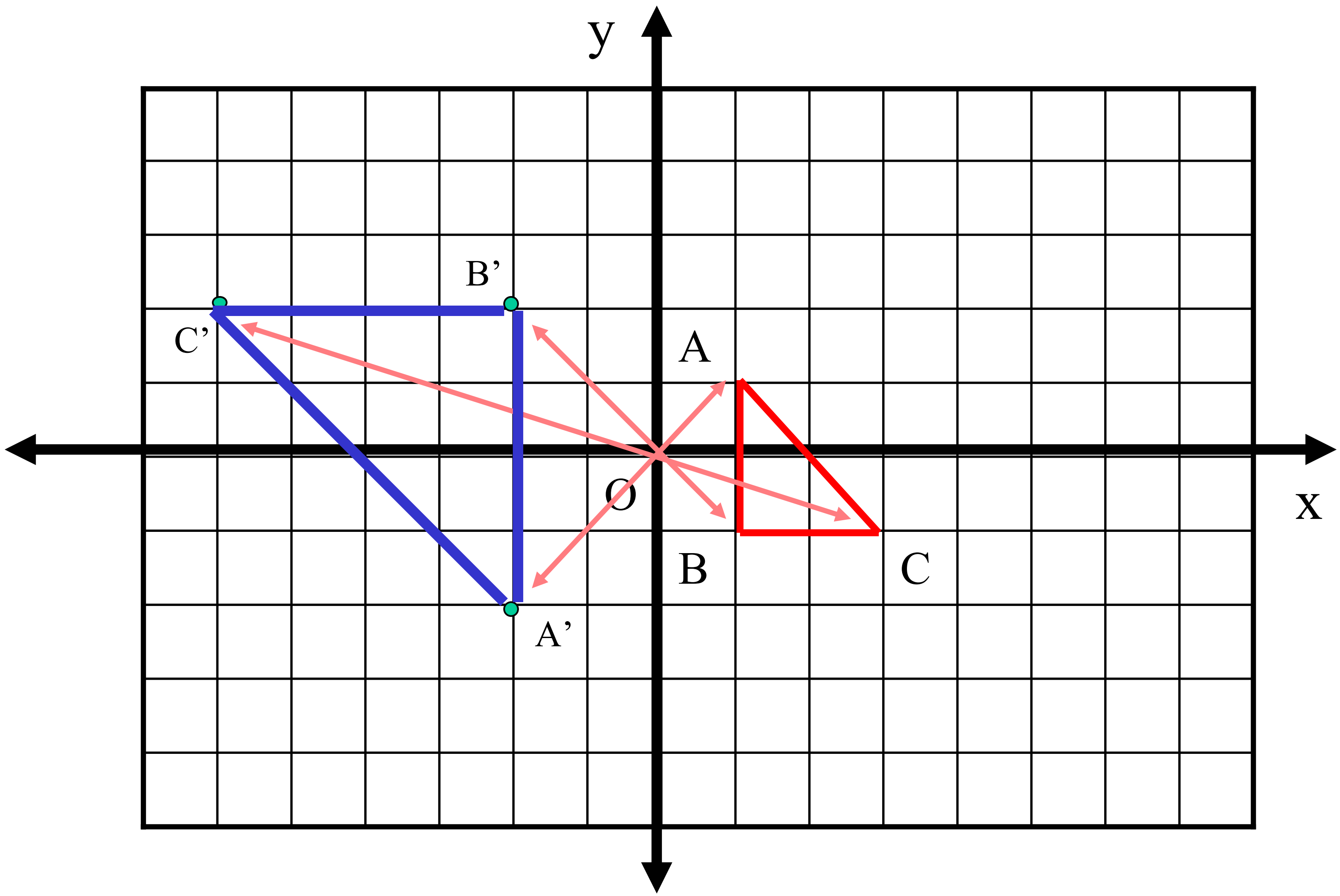
$k = \text{scale factor} = 1/3$

$D_{O,\,1/3}\,(3, 9) \rightarrow (1, 3)$

$|k| < 1$, contraction

Be Careful!

$D_{O,\,-2}\ \Delta ABC$



Sample Problems: Dilation with center NOT the origin.

$D_{A,\,2}\, B \rightarrow B'$

A (2, 1)   B (3, 3)

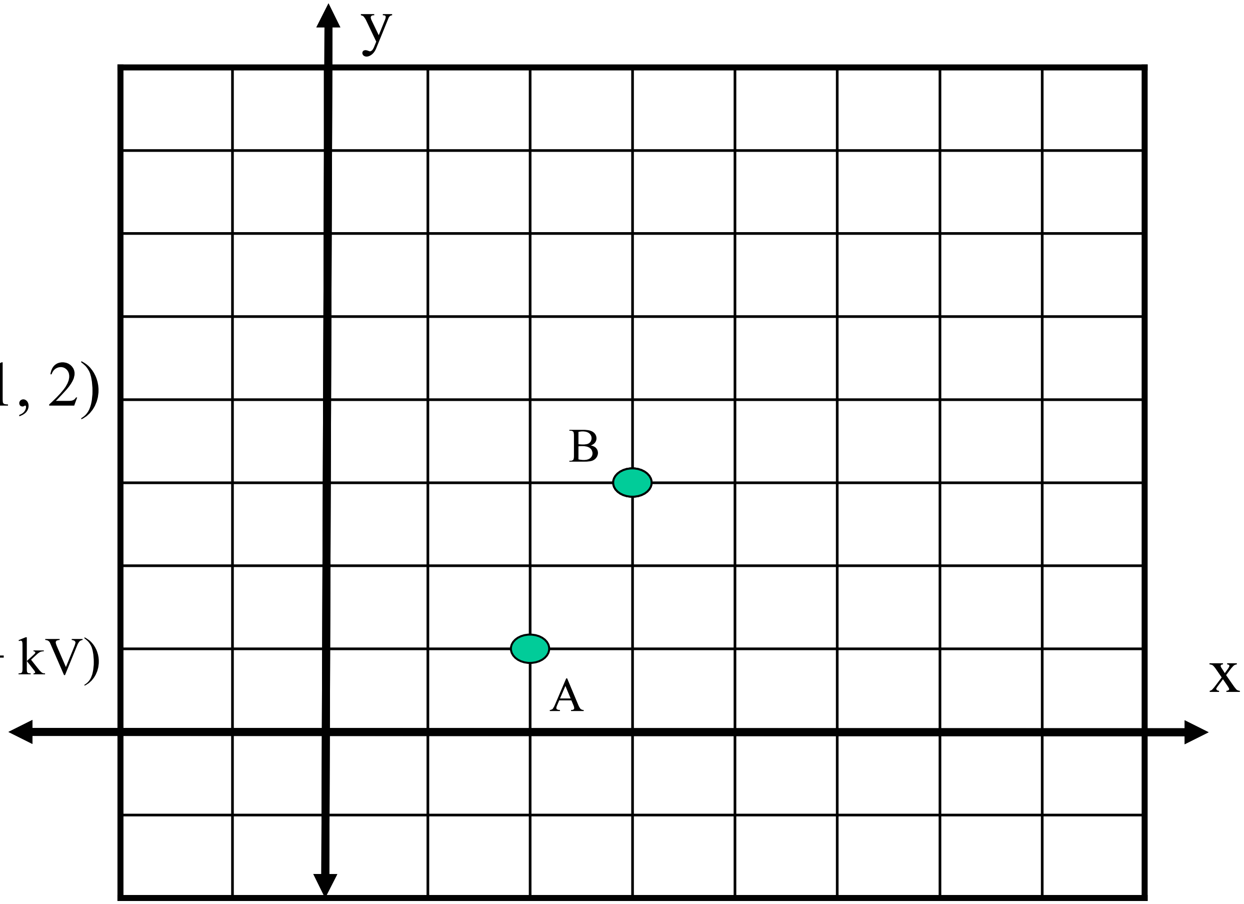
Determine $\overrightarrow{AB}$.

$\overrightarrow{AB} = (3 - 2, 3 - 1) = (1, 2)$

Rewrite $D_{A,\,2}\, B \rightarrow B'$

in form

$D_{O,\,k}\,(x, y) \rightarrow (x + kH, y + kV)$



Substitute x and y coordinates of A into formula.

$D_{A,\,k}\,(2, 1) \rightarrow (2 + kH, 1 + kV)$

Sample Problems: Dilation with center NOT the origin.

$D_{A,2} B \rightarrow B'$

A (2, 1)   B (3, 3)

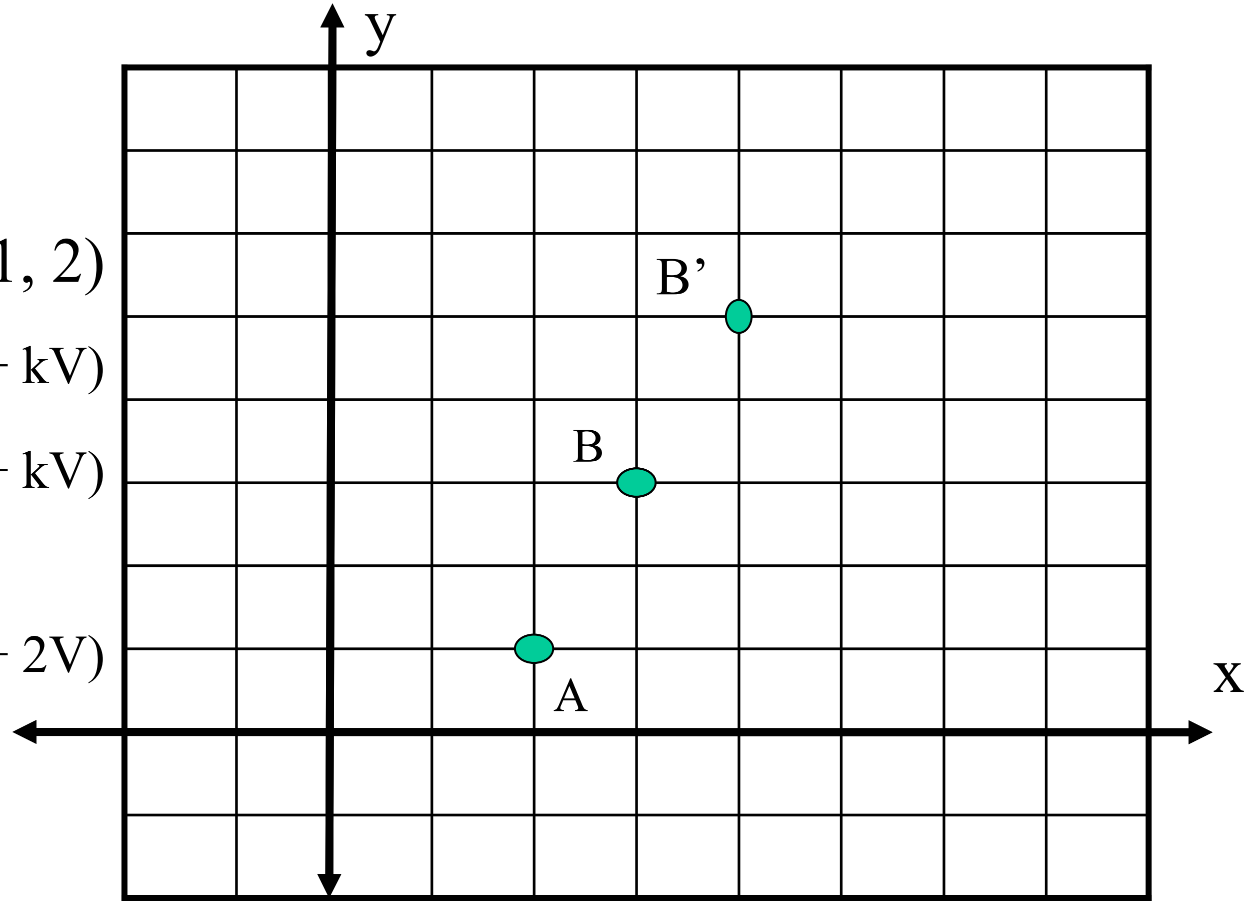
$\overrightarrow{AB} = (3 - 2, 3 - 1) = (1, 2)$

$D_{O,k}(x, y) \rightarrow (x + kH, y + kV)$

$D_{A,k}(2, 1) \rightarrow (2 + kH, 1 + kV)$

Substitute k into formula.

$D_{A,2}(2, 1) \rightarrow (2 + 2H, 1 + 2V)$



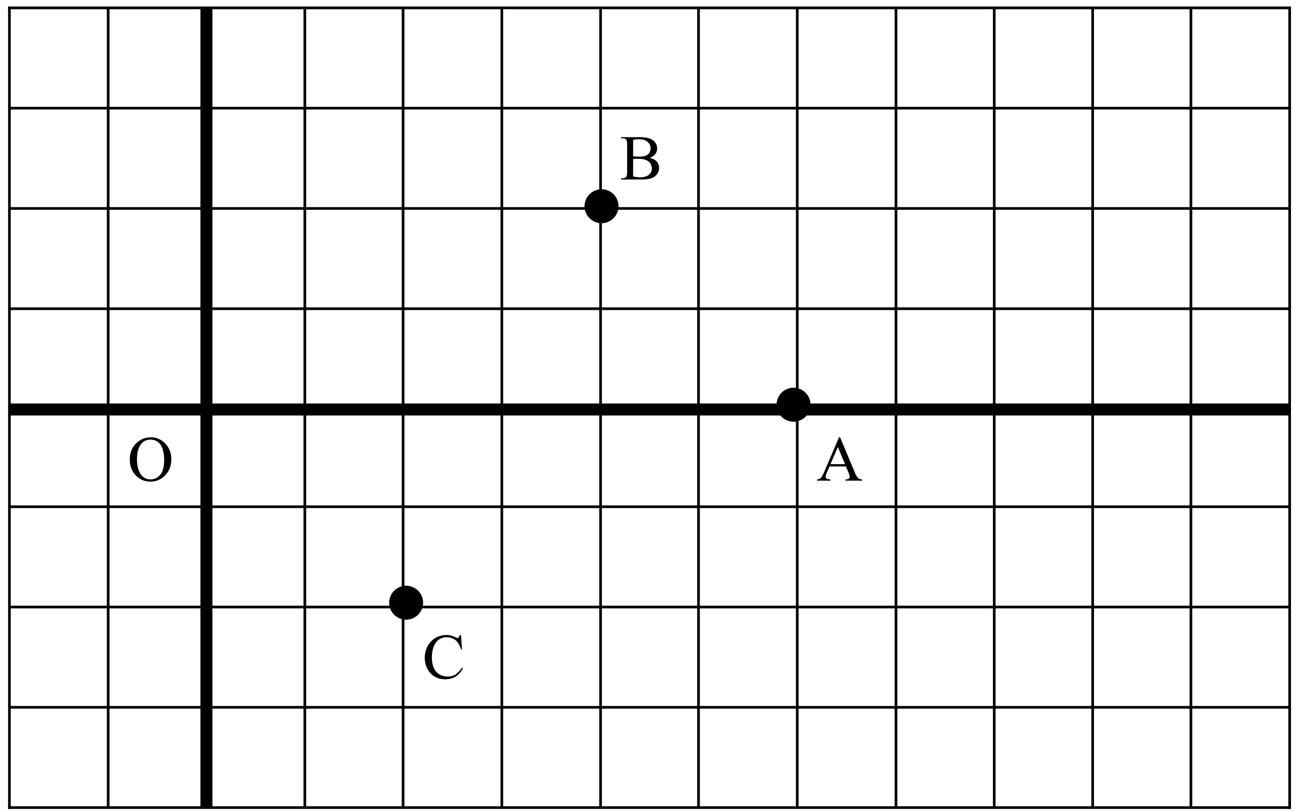
Substitute $\overrightarrow{AB}$ into formula. $D_{A,2}(2, 1) \rightarrow (2 + 2(1), 1 + 2(2))$

Plot point B' (4, 5)

Sample Problems

Find the images of A, B, & C under the given mappings.

1. $D_{O,\,2}$
3. $D_{O,\,½}$
5. $D_{O,\,-2}$
7. $D_{A,\,-½}$



Sample Problems

A dilation with the origin as its center maps the given point to the given image. Find the scale factor of the dilation. Is the dilation an expansion or a contraction?

13. $\left(0, \frac{1}{6}\right) \rightarrow \left(0, \frac{2}{3}\right)$

15. Which of the following properties remain invariant under any dilation? a) distance b) angle measure c) area d) orientation

Sample Problems

Graph quad. PQRS and its image under the dilation given. Find the ratio of the perimeters and the ratio of the areas of the two quadrilaterals.

19. P(12, 0) Q(0, 15) R(- 9, 6) S(3, - 9) $D_{O, 2/3}$

21. P(- 2, - 2) Q(0, 0) R(4, 0) S(6, - 2) $D_{O, -1/2}$

23. A dilation with a scale factor ¾ maps a sphere with center C to a concentric sphere. a) What is the ratio of the surface areas of these spheres? b) What is the ratio of the volumes of these spheres?

Section 14-6

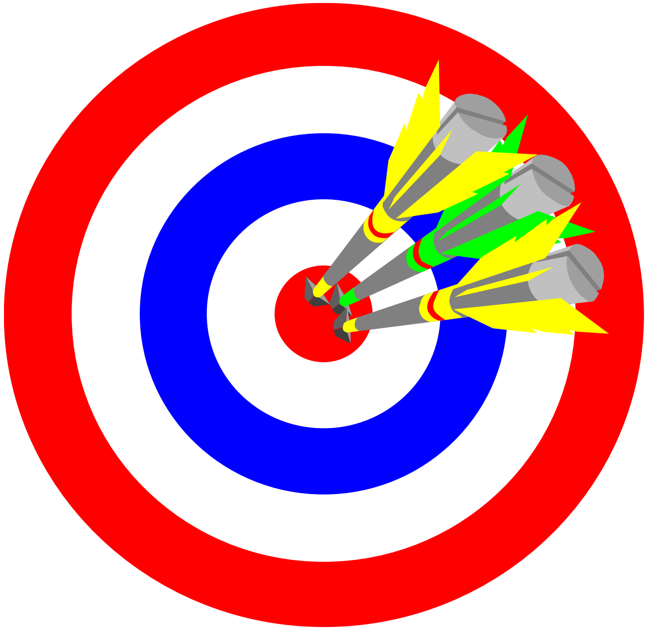
Composites of Mappings

Homework Pages 603-604:

2-24 evens

Excluding 14, 16

Objectives



A. Locate the images of figures by composites of mappings.

- ★ composite: a new mapping formed by combining two or more mappings. Written as
  - function notation: $f(g(x))$
  - mapping notation: $f \circ g: x \rightarrow x'$

To Solve a Composite Function

Let the functions *f* and *g* be defined as:

$f(x) = x^2$ and $g(x) = 2x – 7$. Find $g(f(2))$.

Realize that the composite function $g(f(2))$ is equivalent to the composite mapping $g \circ f: 2$.

$g(f(2))$ states to plug 2 in for x in the function $g(f(x))$.

$g \circ f: 2$ states to use 2 as the pre-image for the composite mapping $g \circ f: x \rightarrow x'$.

To Solve a Composite Function

Let the functions *f* and *g* be defined as:

$f(x) = x^2$ and $g(x) = 2x - 7$. Find $g(f(2))$.

Always start with the right-most or inner-most function or mapping.

In this case $f(2)$ is the inner-most function. Solve for $f(2)$.

$f(2) = x^2 = (2)^2 = 4$

Use this result (4) as the input to the next right-most or inner-most function or mapping.

In this case, the next right-most function is $g(x)$.

Solve $g(x)$ using the result/output (4) of the first function.

$g(4) = 2x - 7 = 2(4) - 7 = 1$

Therefore, $g(f(2)) = 1$.

To Solve a Composite Mapping

Let the mappings *h* and *k* be defined as:

h: (x, y) → (x + 2, y – 1), k: (x, y) → (x – 4, y – 2).

Find h ∘ k: (3, 4)

Always start with the right-most or inner-most function or mapping. In this case k: (x, y) is the right-most mapping. Solve for k: (3, 4). k: (3, 4) → (3 – 4, 4 – 2) = (-1, 2).

Use this result (-1, 2) as the input to the next right-most or inner-most function or mapping.

In this case, the next right-most mapping is h: (x, y).

Solve h: (x, y) using the result/output (-1, 2) of the 1^{st} mapping.

h: (-1, 2) → (-1 + 2, 2 – 1) → (1, 1)

Therefore, h ∘ k: (3, 4) → (1, 1)

Theorem 14-6

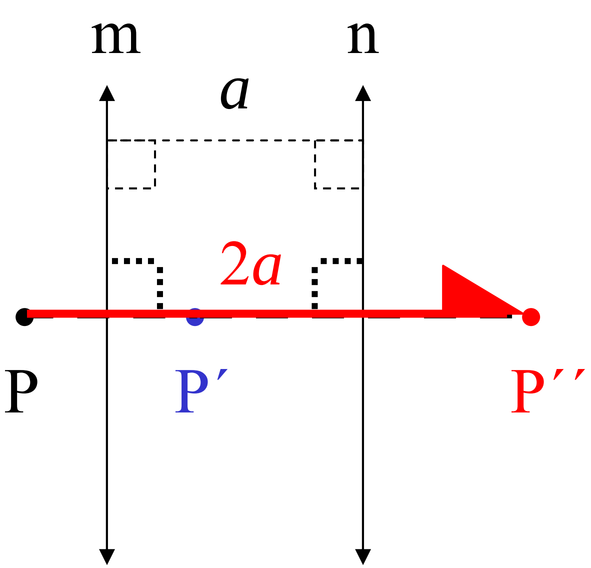
The composite of two isometries is an isometry.

If mappings F & G are one-to-one mappings of the whole plane that preserve distance, then $F \circ G$ and $G \circ F$ are one-to-one mappings of the whole plane that preserve distance .

Theorem 14-7

A composite of reflections in two parallel lines is a translation. The translation glides all points through twice the distance from the first line of reflection to the second.

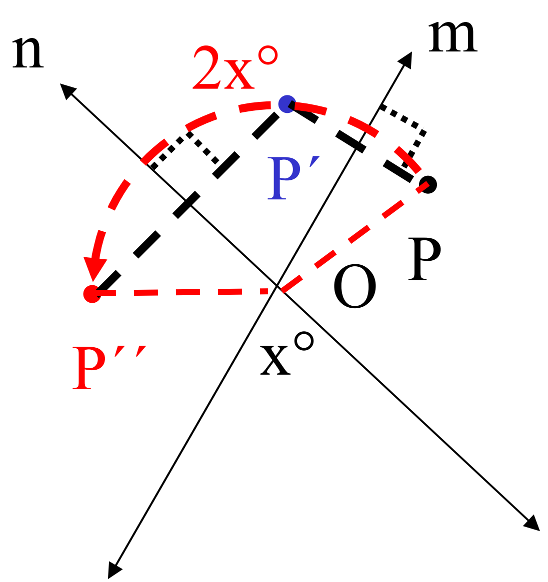
If m $||$ n and the distance from m to n is a, then $R_m \circ R_n$ or $R_n \circ R_m = T: (x, y) \rightarrow (x + H, y + V)$ where vector $PP'' = (H, V)$ & $|PP''| = 2a$



Theorem 14-8

A composite of reflections into intersecting lines is a rotation about the point of intersection of the two lines. The measure of the angle of rotation is twice the measure of the angle from the first line of reflection to the second.

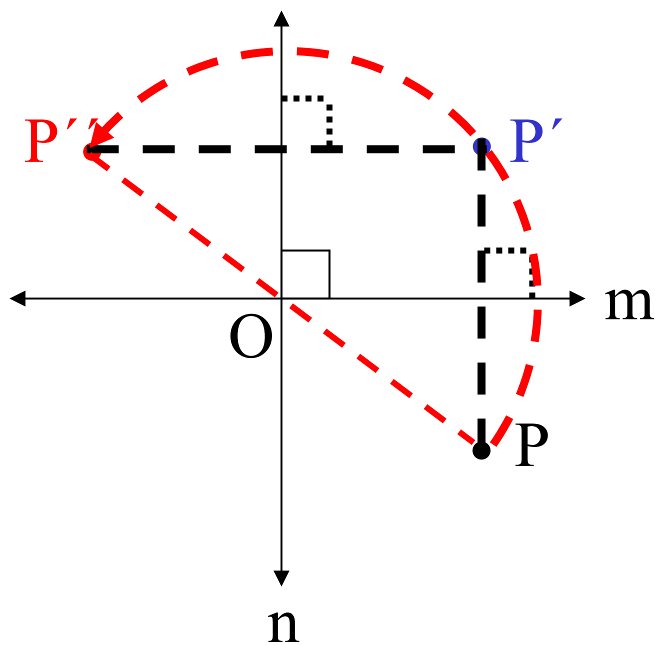
If m & n intersect at O with an angle x°,
then $R_m \circ R_n$ or $R_n \circ R_m = \Re_{O,\, 2x°}$



Theorem 14-8 Corollary 1

A composite of reflections into perpendicular lines is a half-turn about the point where the lines intersect.

If $m \perp n$ and the lines intersect at O,
then $R_m \circ R_n$ or $R_n \circ R_m = H_O$



Sample Problems

11. L, M and N are the midpoints of the sides of the triangle.

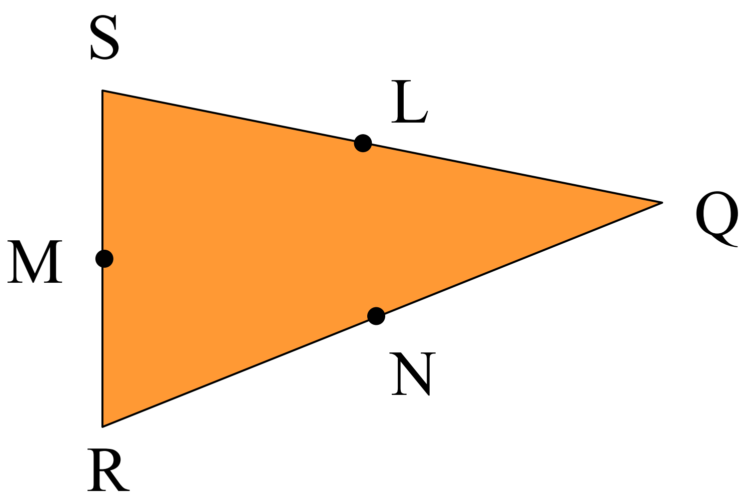
a) $H_N \circ H_M$: S → ?

b) $H_M \circ H_N$: Q → ?

c) $D_{S,½} \circ H_N$: Q → ?

d) $H_N \circ D_{S,2}$: M → ?

e) $H_L \circ H_M \circ H_N$: Q → ?



Start with right-most mapping H_N: Q → ?

A half turn of point Q with center N yields point R.

Use the output of mapping H as input to mapping $D_{S,½}$

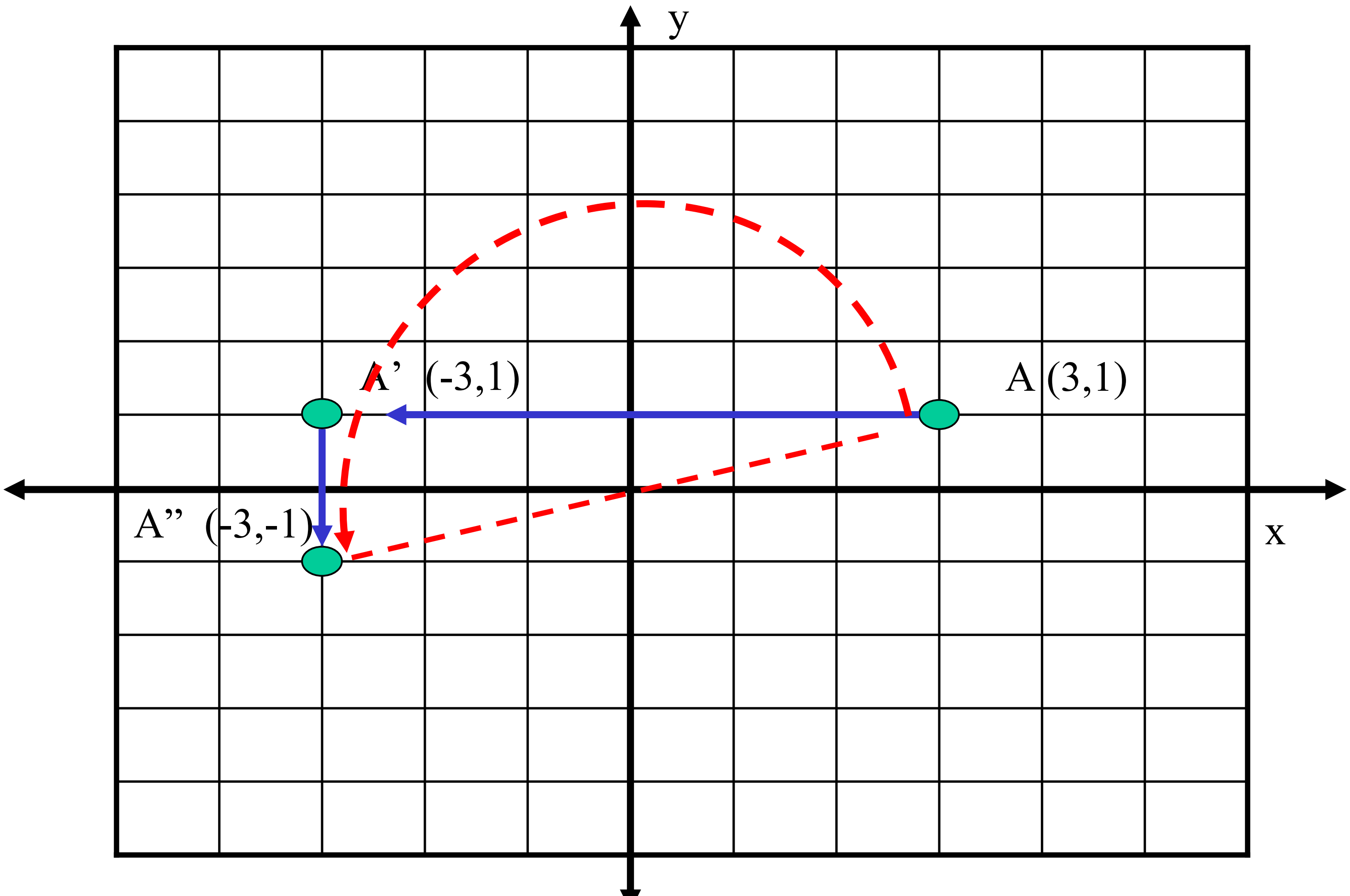
$D_{S,½}$ R → M (remember, M is the midpoint of segment SR).

Therefore $D_{S,½} \circ H_N$: Q → M.

Sample Problems

O is the origin and A(3, 1).

17. $R_x \circ R_y$: (3, 1) → (?, ?)



Sample Problems

1. If $f(x) = x^2$ and $g(x) = 2x - 7$

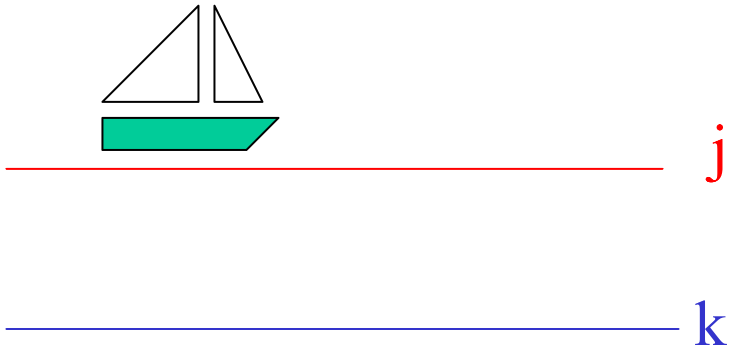
   a) $(g \circ f)(2)$ b) $(g \circ f)(x)$ c) $(f \circ g)(2)$ d) $(f \circ g)(x)$

3. If $h : x \rightarrow \frac{x+1}{2}$ and $k : x \rightarrow x^3$, complete the following

   a) $k \circ h: 3 \rightarrow ?$ b) $k \circ h: 5 \rightarrow ?$ c) $k \circ h: x \rightarrow ?$

   d) $h \circ k: 3 \rightarrow ?$ e) $h \circ k: 5 \rightarrow ?$ f) $h \circ k: x \rightarrow ?$

Copy each figure and find its image under $R_k \circ R_{j}$. Then copy the figure again and find its image under $R_j \circ R_k$.



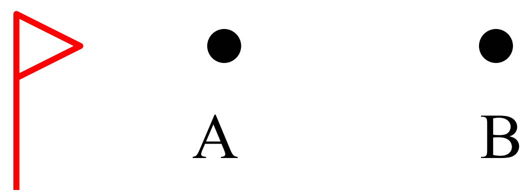
Sample Problems

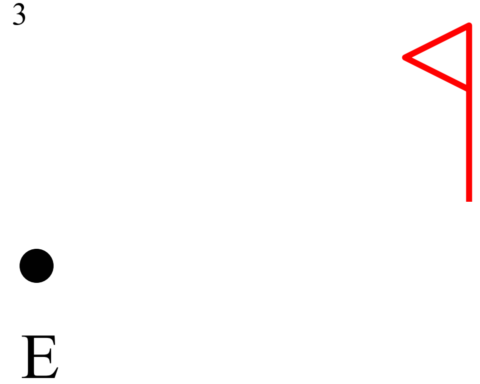
7. a) $H_B \circ H_A$

   b) $H_A \circ H_B$

9. a) $H_E \circ D_{E,\frac{1}{3}}$

   b) $D_{E,\frac{1}{3}} \circ H_E$





Sample Problems

11. L, M and N are the midpoints of the sides of the triangle.

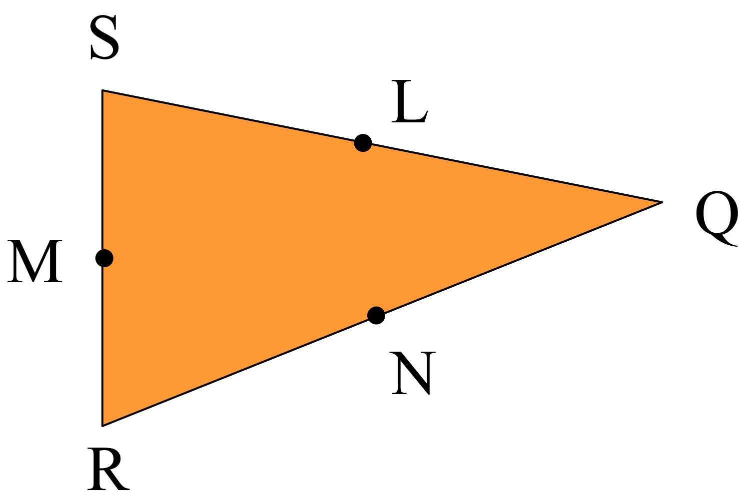
a) $H_N \circ H_M: S \rightarrow ?$

b) $H_M \circ H_N: Q \rightarrow ?$

c) $D_{S,½} \circ H_N: Q \rightarrow ?$

d) $H_N \circ D_{S,2}: M \rightarrow ?$

e) $H_L \circ H_M \circ H_N: Q \rightarrow ?$



Which of the following properties are invariant under the mapping a) distance b) angle measure c) area d) orientation

13. composite of a reflection and a dilation

15. composite of a rotation and a translation

Sample Problems

O is the origin and A(3, 1).

19. $H_A \circ H_O$: (3, 0) $\rightarrow$ (?, ?)

21. $R_x \circ D_{O,2}$: (2, 4) $\rightarrow$ (?, ?)

23. $\mathfrak{R}_{A,90} \circ \mathfrak{R}_{O,-90}$: (- 1, - 1) $\rightarrow$ (?, ?)

25. Let R_l be the reflection into the line y = x. P(5, 2)
    a) Find $R_y \circ R_l$: P
    b) m $\angle$ POQ = ?
    c) verify OP $\perp$ OQ
    d) Find $R_y \circ R_l$: (x, y) and $R_l \circ R_y$: (x, y)

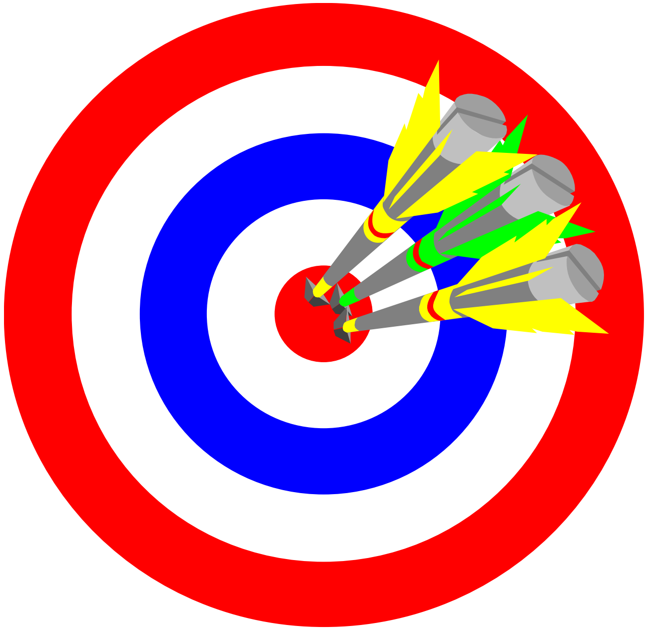
Section 14-7

Inverses and Identities

Homework Pages 607-608:

2-26 evens

Objectives



A. Recognize and use the terms identity and inverse in relation to mappings.

- ★ identity mapping: the mapping that maps every point to itself, called the I mapping.
- ★ inverse: the inverse of a mapping S is the mapping which moves the images of S back to the preimages of S, called S^{-1}.
  - For all mappings $S \circ S^{-1} = I$
- ★ product: the product of a mapping is the result of a composite mapping.
  - $S \circ T$ is the *product* of mappings S and T
  - S^4 is equivalent to the composite mapping $S \circ S \circ S \circ S$

Identity Mapping

- Examples of Identity mappings:
  - Transformation:
    - T: (x, y) → (x, y)
  - Rotation:
    - $\Re_{A,0}$
  - Reflection:
    - R_m: (x, y) where the points (x, y) are on the line *m*
  - Translation:
    - T: (x, y) → (x + H, y + V) where the vector is (0, 0)

Inverse Mapping

- To determine the inverse mapping T^{-1} of a mapping T:
  - Apply the original mapping T to the point, points, or figure.
  - Determine the mapping T^{-1} that will return each point, all points, or the figure to its original position.
- In other words:
  - The pre-image of the original mapping T is the image of the inverse mapping T^{-1}.
  - The image of the original mapping T is the pre-image of the inverse mapping T^{-1}.

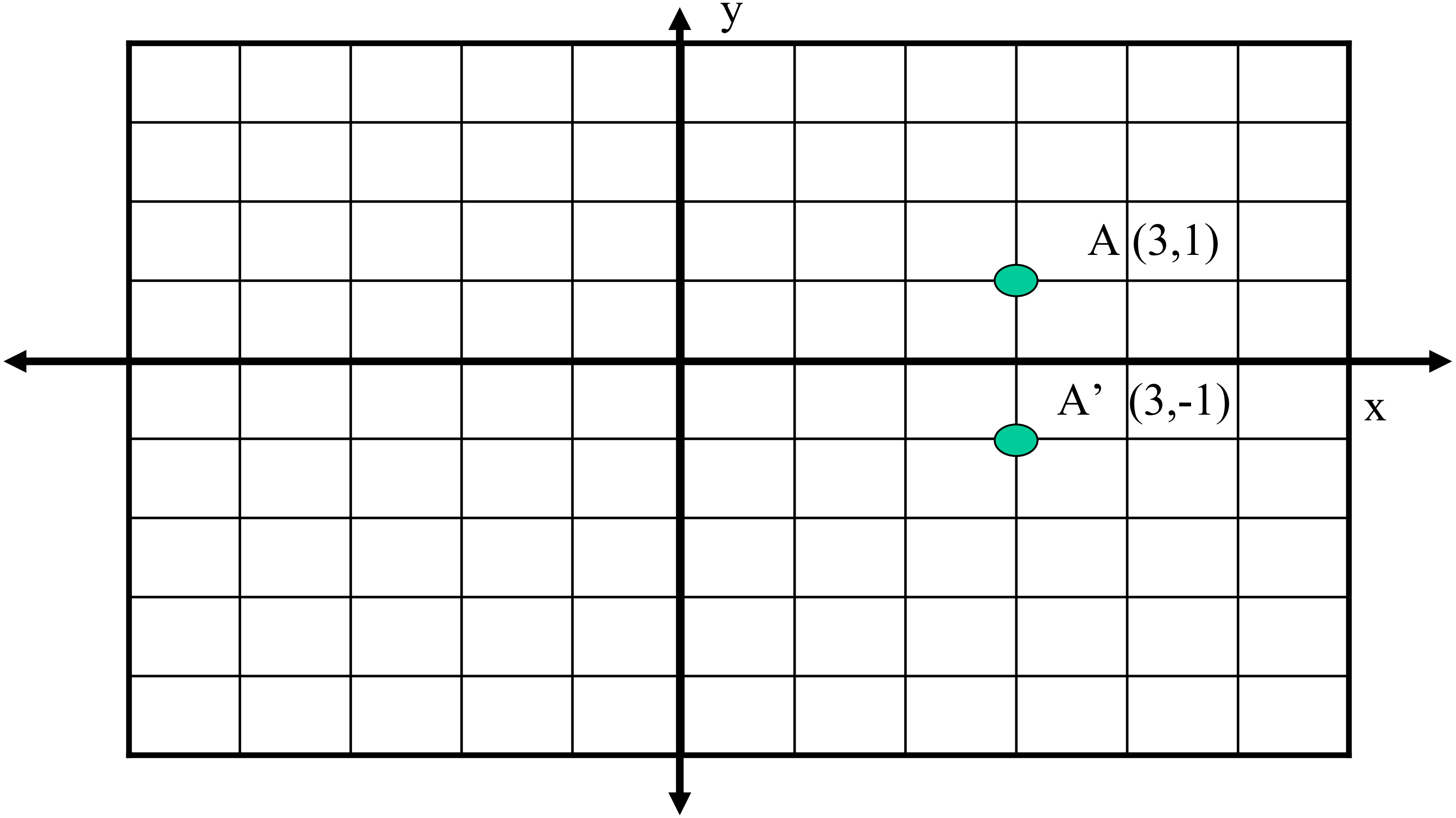
Inverse Mapping Examples

Find the inverse mapping R^{-1} of the mapping R_x: (x, y).

Plot a random point A. Reflect the point over the x-axis.

Determine what reflection will move A' back to A.

R^{-1}_x: (x, y) is the inverse mapping of R_x: (x, y).



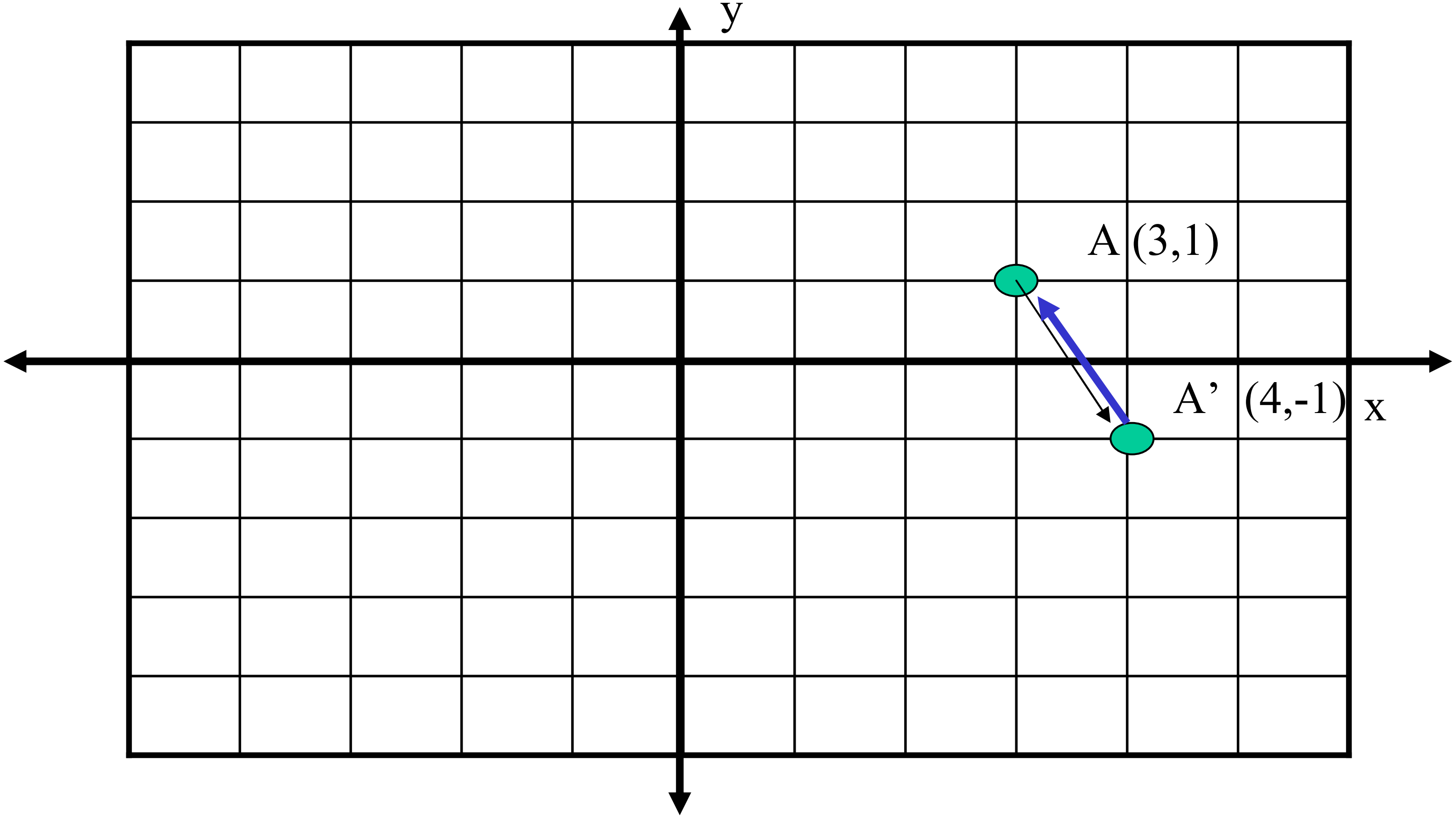
Inverse Mapping Examples

Find the inverse mapping T^{-1} of the mapping T: (x, y) → (x + 1, y – 2).

Plot a random point A. Translate the point.

Determine what translation will move A' back to A.

T^{-1}: (x, y) → (x – 1, y + 2) is the inverse mapping of T.



Product Mapping Example

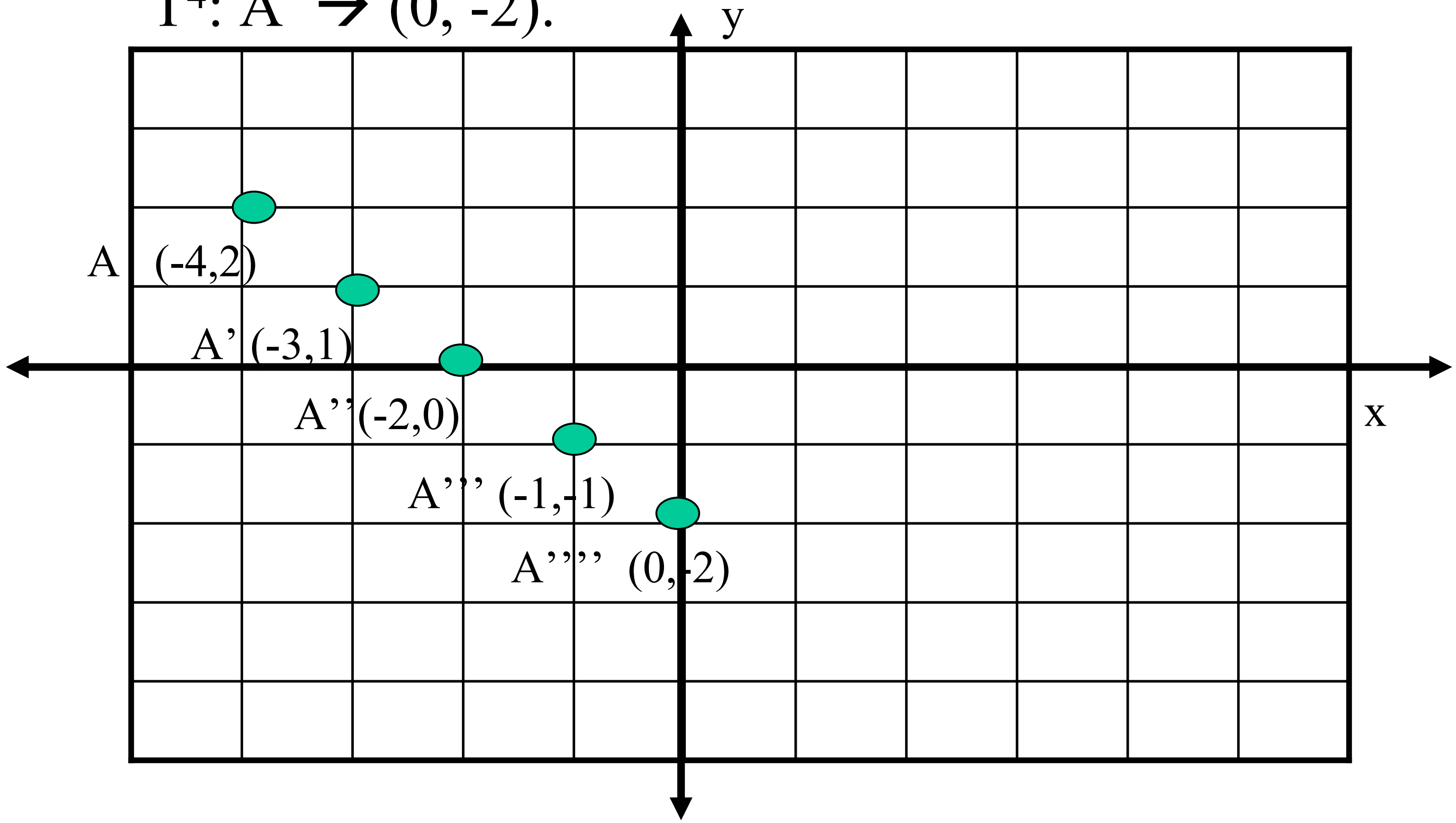
Let T: (x, y) → (x + 1, y – 1) and A (-4, 2). Find T^4:A.

Plot point A. Find and plot point A'. T (-4, 2) → (-3, 1).

Find and plot point A'' using A' as input. T (-3, 1) → (-2, 0).

Repeat process to find A''' and A''''. T (-2, 0) → (-1, -1).

T^4: A → (0, -2).

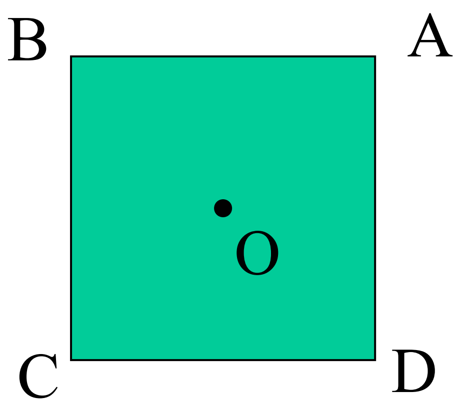


Sample Problems

1. 4^{-1}

3. $\left(\frac{2}{3}\right)^{-1}$

The rotation $\Re$ maps all points 90° about O, the center of the square ABCD. Give the image of A under each mapping.



5. $\Re^{2}$
7. $\Re^{4}$
9. $\Re^{-2}$
11. $\Re^{-3} \circ \Re^{3}$
13. $\Re^{50}$

15. H_O^{2} is the same as the mapping ____?
17. H_O^{3} is the same as the mapping ____?

Sample Problems

19. If T: $(x, y) \rightarrow (x + 3, y - 4)$, then T^2: $(x, y) \rightarrow (?, ?)$.

Write the rule for S^{-1}.

21. S: $(x, y) \rightarrow (x + 5, y + 2)$
23. S: $(x, y) \rightarrow (3x, -\frac{1}{2}y)$,
25. S: $(x, y) \rightarrow (x - 4, 4y)$,

27. If S: $(x, y) \rightarrow (x + 12, y - 3)$, find a translation T such that $T^6 = S$.

Section 14-8

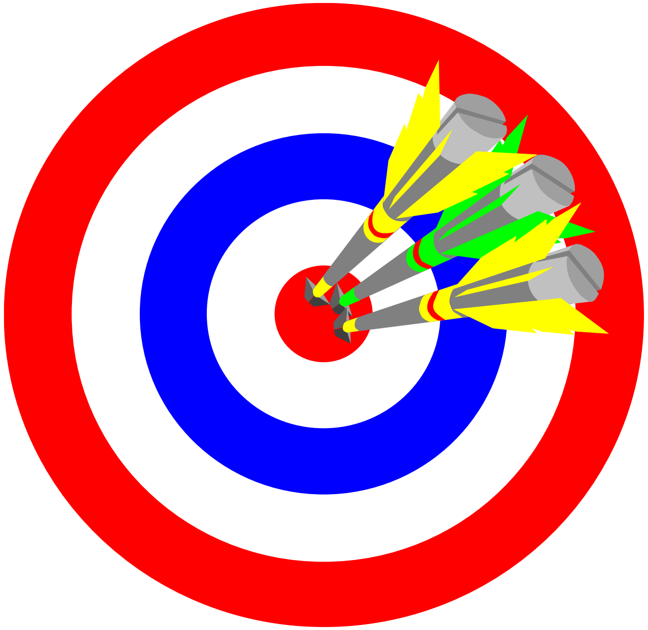
Symmetry in the Plane and in Space

Homework Pages 612-613:

2-22 evens

Excluding 16, 18

Objectives



A. Describe the symmetry of figures and solids.

Symmetry

- symmetry: a figure in a plane has symmetry if there is an isometry, *other than the identity*, that maps the figure onto itself.

Line Symmetry

- line symmetry: the isometry which maps the figure onto itself is a **reflection**. The mirror of the reflection is your **line of symmetry**.

Consider square ABCD. Consider the line $\overleftrightarrow{BD}$

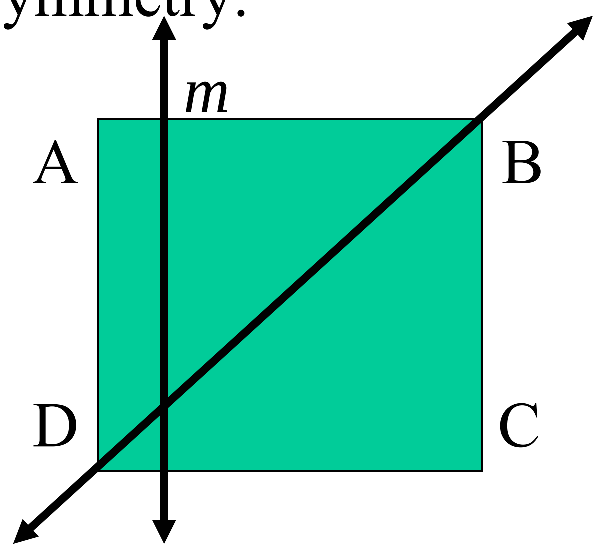
What does the reflection of square ABCD over the line BD produce?

What does the reflection of square ABCD over the line AC produce?

So R_{BD} and R_{AC} are line symmetries of the square ABCD and lines BD and AC are lines of symmetry.

Are there other lines of symmetry?

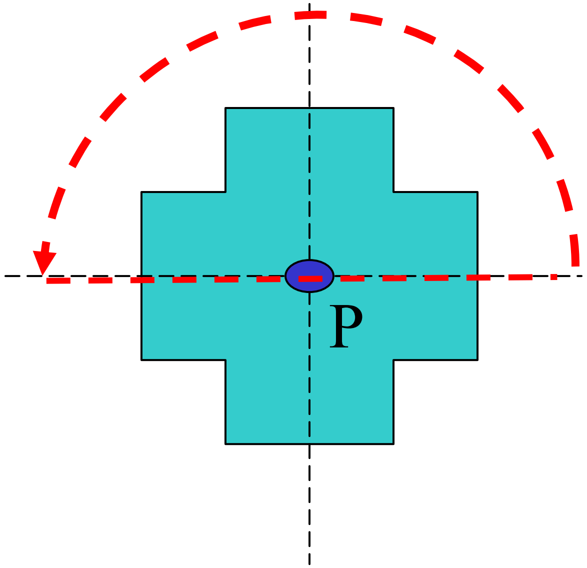
Is line *m* a line of symmetry for square ABCD?



Point Symmetry

- point symmetry: the isometry which maps the figure onto itself is a **half-turn**. The center of the half-turn is the **point of symmetry**

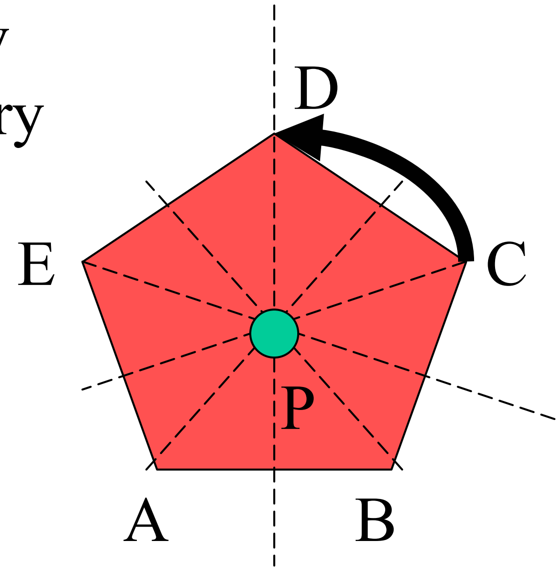
H_P is the point symmetry and P is the point of symmetry.



Rotational Symmetry

- rotational symmetry: the isometry which maps the figure onto itself is a **rotation**. The angle of rotation is the **symmetry angle**. Rotational symmetries can have more than one symmetry angle.

$\mathfrak{R}_{P,72°}$ Pentagon ABCDE is a rotational symmetry with 72° as the symmetry angle.

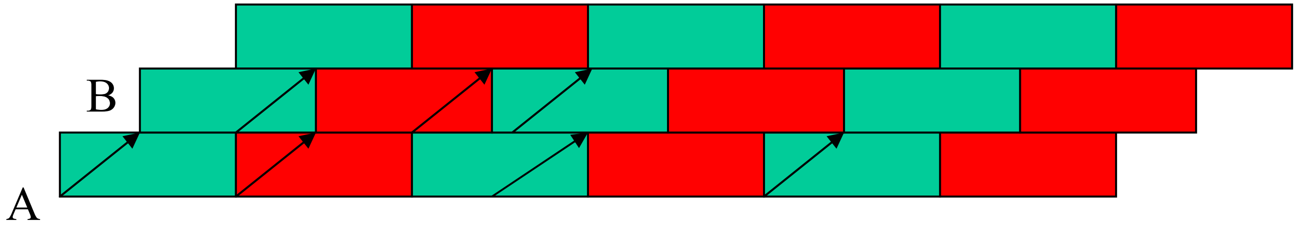


Translational Symmetry

- translational symmetry: the isometry that maps the figure onto itself is a **translation**. The vector for the translation is the **vector of symmetry**. There can be more than one vector of symmetry.

Vector AB = (H, V)

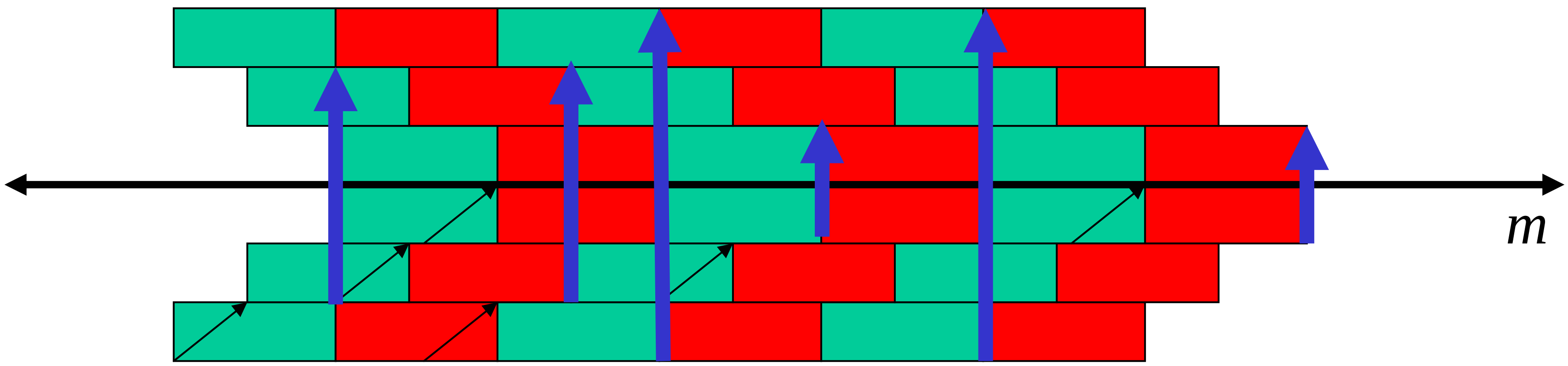
All points on the bottom row follow the same vector to their corresponding point in the second row.



Therefore T: (x, y) → (x + H, y +V) is the translational symmetry and Vector AB is the vector of symmetry.

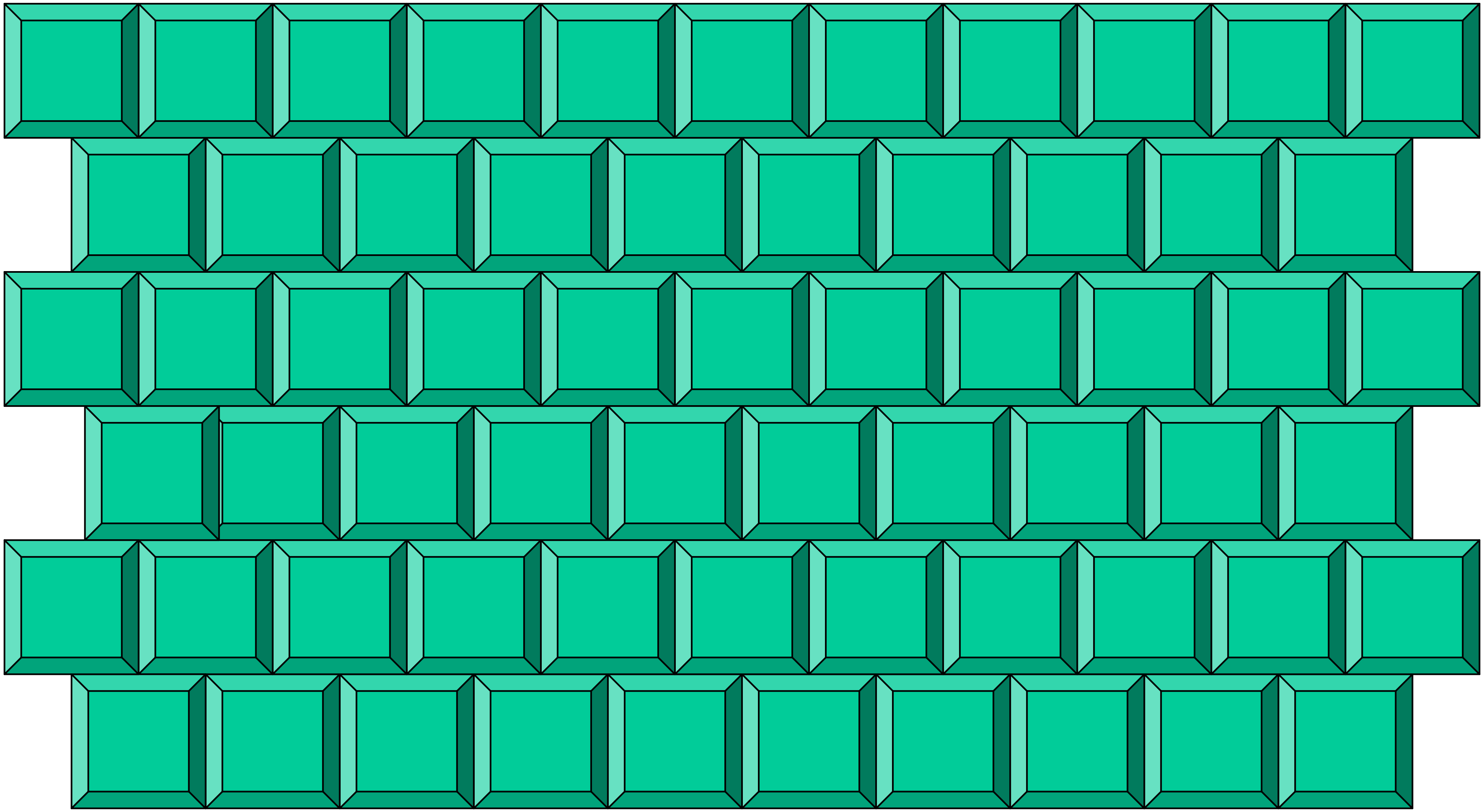
Glide Reflection Symmetry

- glide reflection symmetry: the isometry that maps the figure onto itself is a **glide reflection**. The vector and the mirror of the glide reflection are your vector and your line of symmetry. There can be more than one pair of vector and line of symmetry.



Tessellation

- tessellation: a pattern where congruent copies of a figure fill the plane.



- plane symmetry: the isometry which maps the figure onto itself is a reflection into a plane.
- bilateral symmetry: a figure with a single plane of symmetry

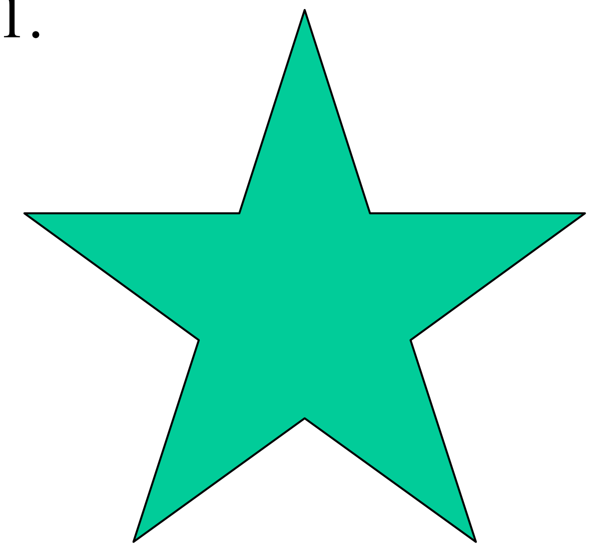
Sample Problems

a) State how many symmetry lines the figure has.

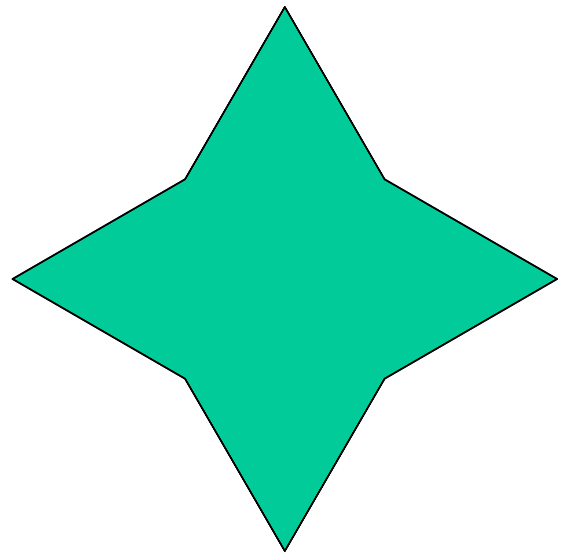
b) State whether the figure has point symmetry.

c) List all rotational symmetries between 0° and 360°.

1.



3.

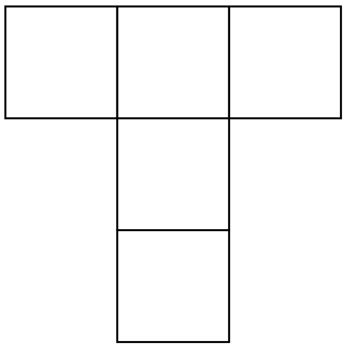


Sample Problems

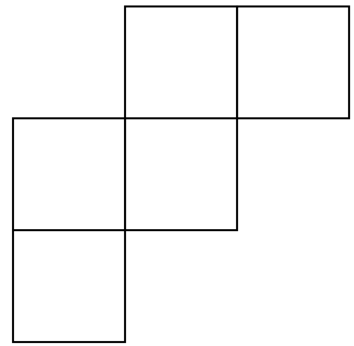
5. Which capital letters of the alphabet have just one line of symmetry.
7. Which capital letters of the alphabet have point symmetry.

Make a tessellation of the given figure.

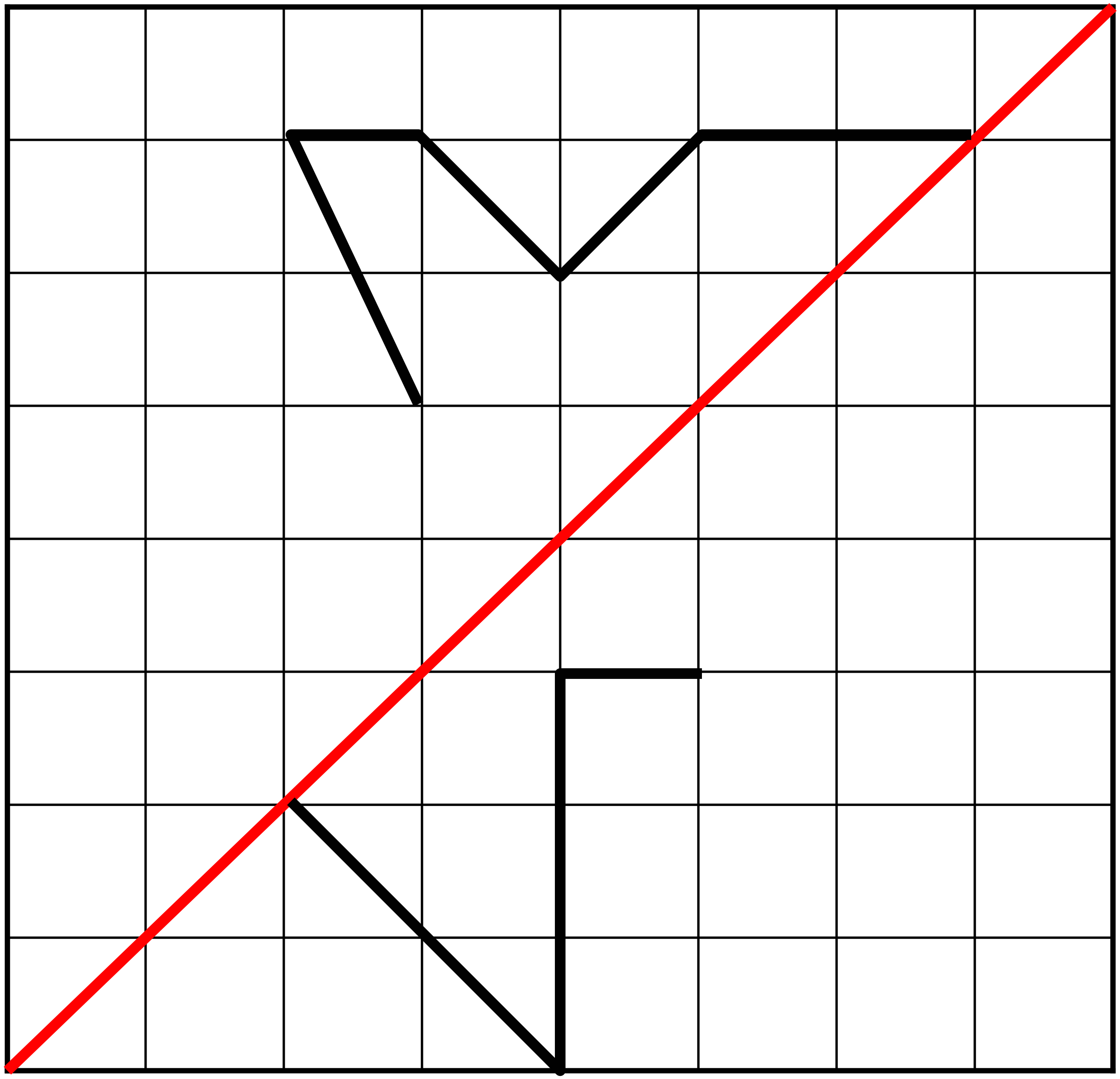
9.



11.



Sample Problems



symmetry in line k

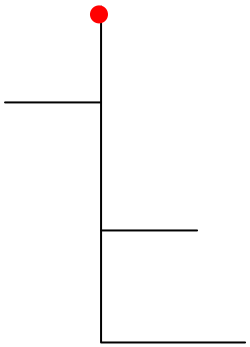
Sample Problems

Complete the figure so that it has the specified symmetry.

15.

60°, 120° and 180° rotational symmetry

17.



2 symmetry lines and 1 symmetry point

Sample Problems

Draw the figure that meets the criteria otherwise write not possible.

21. A trapezoid with a) no symmetry b) one symmetry line c) a symmetry point
23. An octagon with a) eight rotational symmetries b) just four rotational symmetries c) only point symmetry

Chapter 14

Transformations

Review

Homework Page 620: 2-22 evens