Section 7-1

Completing the Square

Objectives

- to solve quadratic equations by completing the square

Quadratic Equations

- standard quadratic form is: $y = ax^2 + bx + c$
- When solving a quadratic equation remember that you are finding the x-intercepts of the graph so that $y = 0$.
- One way to solve an unfactorable quadratic equation is by completing the square.
  - move the constant to one side of the equation
  - divide through by (a) to make the quadratic coefficient 1
  - divide the linear coefficient (b) by 2, square this value and add it to both sides of the equation
  - factor the pattern polynomial that appears on one side of the equal sign and simplify the numerical values that appear on the other side of the equal sign
  - take the square root of both sides (remember to use both the principle and secondary roots since you are solving)
  - isolate the variable

Problems 1-38: Solve

Complete the square.

I. move the constant to one side of the equation

II. divide through by (a) to make the quadratic coefficient 1

III. divide the linear coefficient (b) by 2, square this value and add it to both sides of the equation

IV. factor the pattern polynomial that appears on one side of the equal sign and simplify the numerical values that appear on the other side of the equal sign

V. take the square root of both sides (remember to use both the principle and secondary roots since you are solving)

VI. isolate the variable

Example 23: $5n^2 + 100 = 30n$

I. $5n^2 - 30n = -100$

II. $n^2 - 6n = -20$

III. $-6 \div 2 = -3, (-3)^2 = 9$

$n^2 - 6n + 9 = -20 + 9$

IV. $(n - 3)^2 = -11$

V. $\sqrt{(n-3)^2} = \sqrt{-11}$

VI. $n - 3 = \pm i\sqrt{11}$

VII. $n = 3 \pm i\sqrt{11}$

Section 7-2

The Quadratic Formula

Objectives

- to solve quadratic equations with the quadratic formula

Quadratic Formula

- If you were to complete the square on the formula: $0 = ax^2 + bx + c$ you would get

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Problems 1-40: Solve using the quadratic formula.

I. arrange into $0 = ax^2 + bx + c$ form

II. identify the values of a, b & c and substitute into the quadratic formula

III. simplify

Example 17: $\dfrac{2m^2+16}{5} = 2m$

I. $2m^2 + 16 = 10m$

$2m^2 - 10m + 16 = 0$

II. $a = 2,\ b = -10,\ c = 16$

III. $$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(2)(16)}}{2(2)}$$

$$x = \frac{10 \pm \sqrt{100 - 128}}{4}$$

$$x = \frac{10 \pm \sqrt{-28}}{4}$$

$$x = \frac{10 \pm 2i\sqrt{7}}{4}$$

$$x = \frac{5 \pm i\sqrt{7}}{2} \text{ or } \frac{5}{2} \pm \frac{i\sqrt{7}}{2}$$

Section 7-3

The Discriminant

Objectives

- to determine the nature of the roots of a quadratic equation by using the discriminant
- to solve quadratic equations by the best method
- to solve for a missing coefficient or constant so that the quadratic equation has a specified type of root

The Discriminant

- the discriminant $D = b^2 - 4ac$ will determine the nature of the roots of a quadratic equation in the form $y = ax^2 + bx + c$ only if a, b & c are all real numbers
- $D > 0$, then two unequal real roots
- $D = 0$, then one real double root
- $D < 0$, then two conjugate imaginary roots
- D is a perfect square and a, b & c are integers then roots are rational.

Problems 1-12: Determine the nature of the roots.

I. rearrange the equation into standard quadratic form

II. calculate the value of the discriminant

III. determine the nature of the roots based on the previous notes

Example 9: $d^2 + \frac{7}{3}d = 2$

I. $3d^2 + 7d = 6$

$3d^2 + 7d - 6 = 0$

II. $D = (7)^2 - 4(3)(-6)$

$D = 49 + 72$

$D = 121$

III. real, unequal, rational

Problems 31-36: find the value(s) of k for which each equation has the following: (a) real double root (b) two different real roots and (c) imaginary roots

I. rearrange into standard quadratic form

II. Identify the values of a, b & c

III. (a) solve $D = 0$
(b) solve $D > 0$ write answer as a number line
(c) solve $D < 0$ write answer as a number line

Example 33: $k^2x^2 - 8x + 4 = 0$

I. it already is in standard quadratic form

II. $a = k^2$, $b = -8$ $c = 4$

III. (a) $64 - 16k^2 = 0$

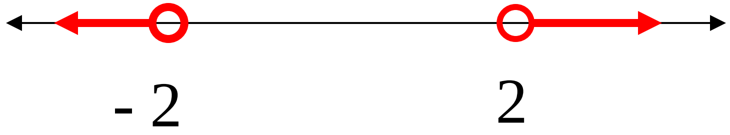
$k = \pm 2$

(b) $64 - 16k^2 > 0$

$(2 - k)(2 + k) > 0$

(c) $64 - 16k^2 < 0$

$(2 - k)(2 + k) < 0$



Section 7-4

Equations in Quadratic Form

Objectives

- to solve non-quadratic equations that have the basic standard quadratic form

Quadratic Form

- the problem fits one of the 4 pattern quadratic polynomials
- the problem has three terms
  - one term is a constant
  - one term has a variable
  - one term uses the same variable but the exponent has been doubled
- $a(\text{something})^2 + b(\text{something}) + \text{constant} = 0$
- “something can be any variable term or polynomial so long as it is the same for both the quadratic and the linear term

Problems 1-26: Solve

I. move all terms to one side

II. arrange them in decreasing order by degree

III. identify the variable part of the linear term

IV. substitute a different variable in for this quantity in both the linear and the quadratic term

V. factor the resulting equation and solve for the new variable

VI. set those values equal to the quantity that you removed by substitution and solve for the original variable

VII. When the original problem is a radical equation or when there are variables in the denominator don't forget to check for extraneous roots!

Example 1b: $(2x - 1)^2 - 5(2x - 1) + 4 = 0$

I. all the terms are on one side

II. the terms are in decreasing order by degree

III. the variable part of the linear term is the quantity $2x - 1$

IV. $y = 2x - 1$ therefore the problem now becomes
$y^2 - 5y + 4 = 0$

V. $(y - 1)(y - 4) = 0$
$y = 1, 4$

VI. $2x - 1 = 1$ and $2x - 1 = 4$

$x = 1, \frac{5}{2}$

Do not follow the directions or the formula in the book. Identify and plot the 11 characteristics you will learn today for Standard Parabolic Form

Section 7-5

Standard Form Parabolic Equations

Objectives

- to graph parabolic equations

Standard Parabolic Form

- all single variable quadratic equations describe a parabola
- parabolas can open in any direction but we will only be studying simple parabolas that open either up, down, left or right.
- Standard Parabolic Form is based on the definition of a parabola: the set of points equidistant from a given point (called the focus) and a given line (called the directrix). To see this derived look at the appendix for Chapter 7.
  - opens up: $(x - h)^2 = 4p(y - k)$
  - opens down: $(x - h)^2 = - 4p(y - k)$
  - opens right: $(y - k)^2 = 4p(x - h)$
  - opens left: $(y - k)^2 = - 4p(x - h)$

Rearranging into Standard Parabolic Form

- Begin in standard quadratic form.
- If the equation equals zero, then remove the zero and replace with a (y).
- Complete the square.
  - $ax^2 + bx + c = y$
  - move the constant to the side with y.
  - divide by (a) so that the quadratic coefficient is 1
  - divide the linear coefficient by 2, square it, and add it to both sides of the equation.
  - factor the side with the variable into a binomial squared
- Factor out a GCF on the side with (y) so that the coefficient of (y) is 1.

Example of rearranging into Standard Parabolic Form

$$3x^2 - 2x + 1 = 0$$

$$3x^2 - 2x + 1 = y$$

$$3x^2 - 2x = y - 1$$

$$x^2 - \frac{2}{3}x = \frac{1}{3}y - \frac{1}{3}$$

$$\left[\left(-\frac{2}{3}\right)\left(\frac{1}{2}\right)\right]^2 = \left[-\frac{1}{3}\right]^2 = \frac{1}{9}$$

$$x^2 - \frac{2}{3}x + \frac{1}{9} = \frac{1}{3}y - \frac{1}{3} + \frac{1}{9}$$

$$\left(x - \frac{1}{3}\right)^2 = \frac{1}{3}y - \frac{2}{9}$$

$$\left(x - \frac{1}{3}\right)^2 = \frac{1}{3}\left(y - \frac{2}{3}\right)$$

standard parabolic form

Characteristics of a Parabola

1. coordinates of x -intercept(s)
2. coordinates of y-intercept(s)
3. coordinates of vertex
4. opens up, down, left or right
5. value of p
6. coordinates of focus
7. equation of the directrix
8. equation of the axis of symmetry
9. description of the Domain
10. description of the Range
11. maximum or minimum point

Locating and Graphing the Parts of Standard Parabolic Form

- Rearrange the equation into standard parabolic form by completing the square.
- Find and plot the x & y intercepts by substituting in zero.
- Determine if the parabola opens up, down, right or left.
- Identify the vertex, (h, k) and plot on a graph.
- Draw a rough sketch through the intercepts and vertex.
- Solve for (p).
- Find and plot the focus on the graph by moving (p) units from the vertex inside the parabola.
- Find and plot the equation of the directrix on a graph by moving (p) units from the vertex outside the parabola to either a horizontal (opens up/down) or vertical (opens right/left) line.
- Find and plot the equation of the axis of symmetry on the graph by connecting the vertex and the focus.
- From the graph describe the Domain and Range.
- Identify, if any, the maximum or minimum value of the graph.

Finding the Intercepts

- For a parabola the intercept(s) are where the graph crosses the x and y-axis.
- Remember any point on the x-axis has a y-value of 0 and any point on the y-axis has an x value of 0.

Example of locating x-intercept(s)

$$\left(x-\frac{1}{3}\right)^2=\frac{1}{3}\left(y-\frac{2}{3}\right)$$

$$\left(x-\frac{1}{3}\right)^2=\frac{1}{3}\left(0-\frac{2}{3}\right)$$

$$\left(x-\frac{1}{3}\right)^2=\frac{1}{3}\left(-\frac{2}{3}\right)$$

$$\left(x-\frac{1}{3}\right)^2=-\frac{2}{9}$$

$$\sqrt{\left(x-\frac{1}{3}\right)^2}=\sqrt{-\frac{2}{9}}$$

$$x-\frac{1}{3}=\pm\frac{i\sqrt{2}}{3}$$

$$x=\frac{1}{3}\pm\frac{i\sqrt{2}}{3}$$

X-Intercepts are the points:

$$\left(\frac{1}{3}+\frac{i\sqrt{2}}{3},0\right)\ \&\ \left(\frac{1}{3}-\frac{i\sqrt{2}}{3},0\right)$$

The parabola does not cross the x-axis because the intercepts are imaginary.

Example of locating y-intercept(s)

$$\left(x-\frac{1}{3}\right)^2=\frac{1}{3}\left(y-\frac{2}{3}\right)$$

$$\left(0-\frac{1}{3}\right)^2=\frac{1}{3}\left(y-\frac{2}{3}\right)$$

$$\left(-\frac{1}{3}\right)^2=\frac{1}{3}\left(y-\frac{2}{3}\right)$$

$$\frac{1}{9}=\frac{1}{3}\left(y-\frac{2}{3}\right)$$

$$\frac{1}{3}=y-\frac{2}{3}$$

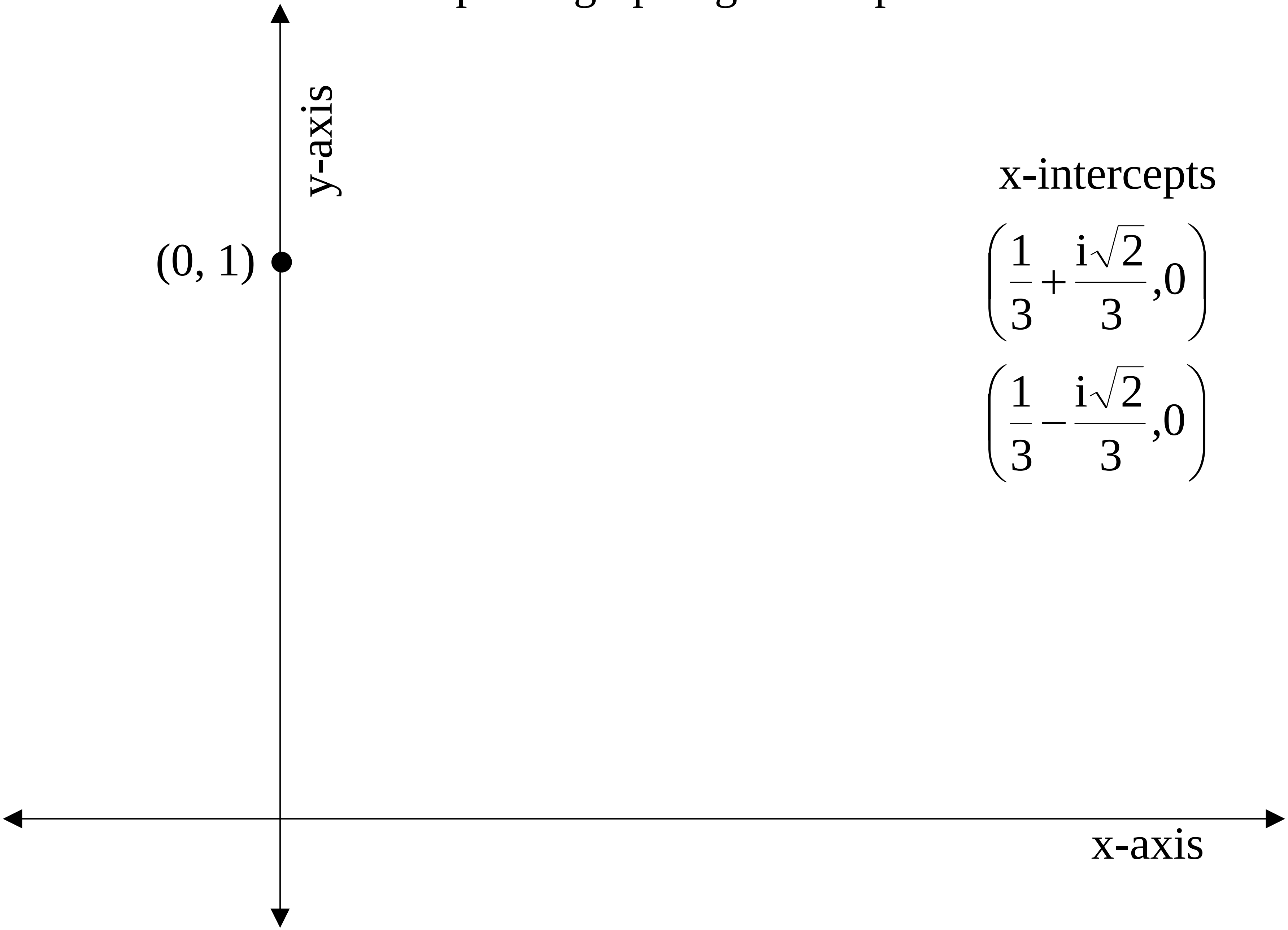
$$y=1$$

y-intercept is the point

$$(0,1)$$

The parabola crosses the y-axis once.

Example of graphing intercepts

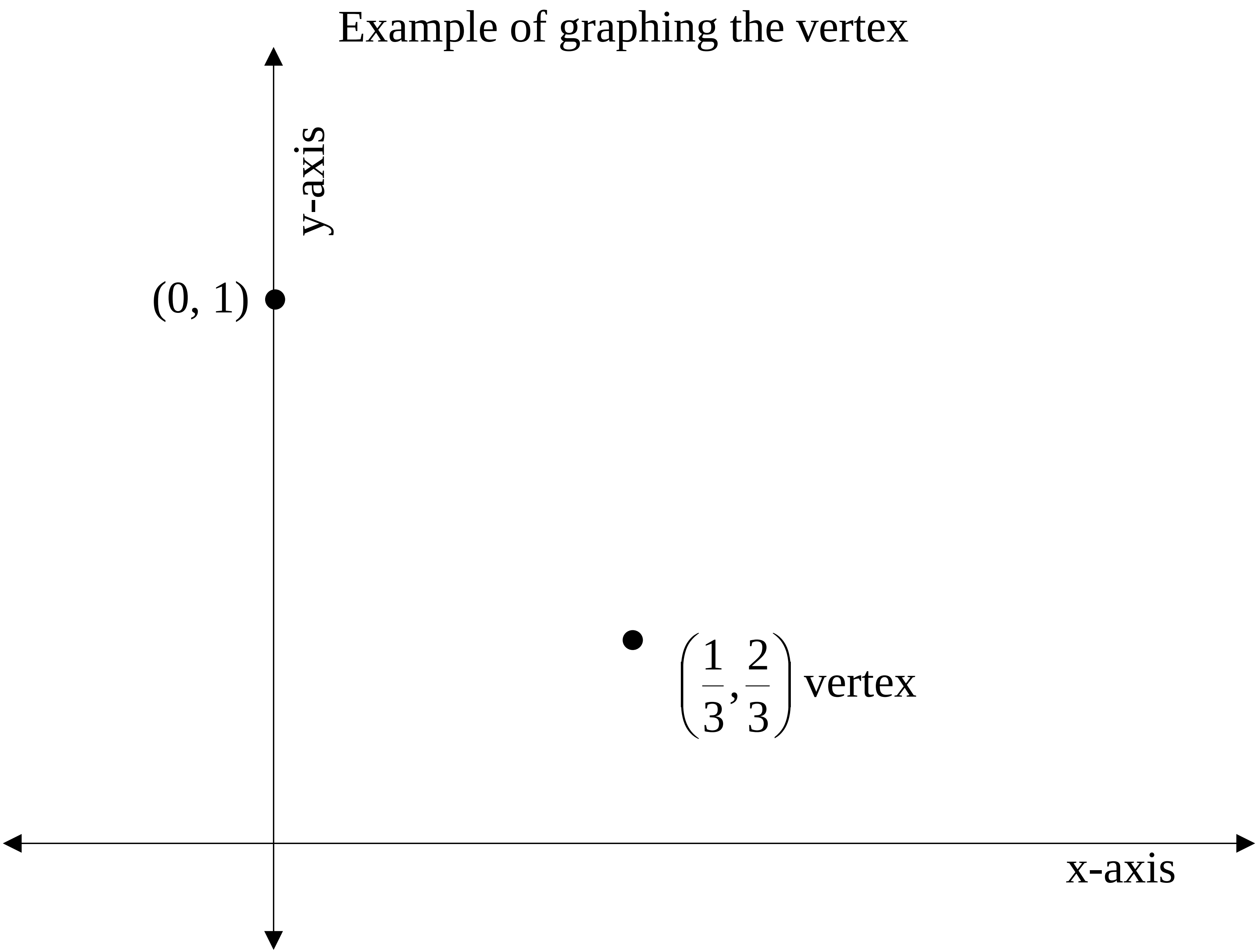


Example of identifying the vertex

$$\left(x-\frac{1}{3}\right)^2=\frac{1}{3}\left(y-\frac{2}{3}\right)$$

$$(x-h)^2=4p(y-k)$$

$$\text{vertex}\left(\frac{1}{3},\frac{2}{3}\right)$$


Example of graphing the vertex
y-axis
(0, 1)
$\left(\frac{1}{3}, \frac{2}{3}\right)$ vertex
x-axis

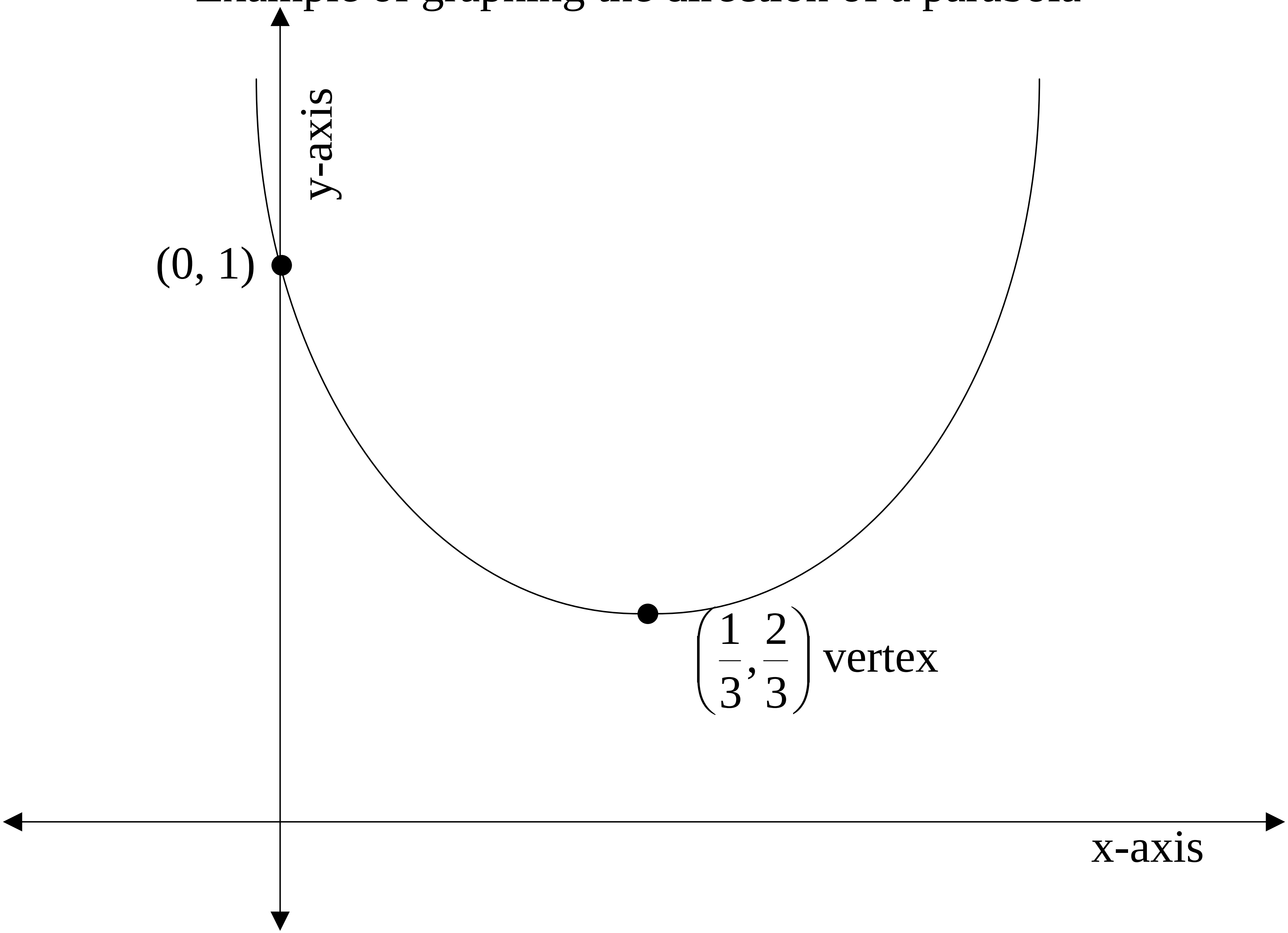
Example of determining the direction of the parabola

- Once it is in standard parabolic form look at the number outside the parentheses and the variable inside the parentheses on the linear side of the equation:

$$\left(x-\frac{1}{3}\right)^2=\frac{1}{3}\left(y-\frac{2}{3}\right)$$

- The number outside the parentheses is positive and the variable inside is y. Because positive y values are “up” on a graph, we know that the parabola opens up.

Example of graphing the direction of a parabola



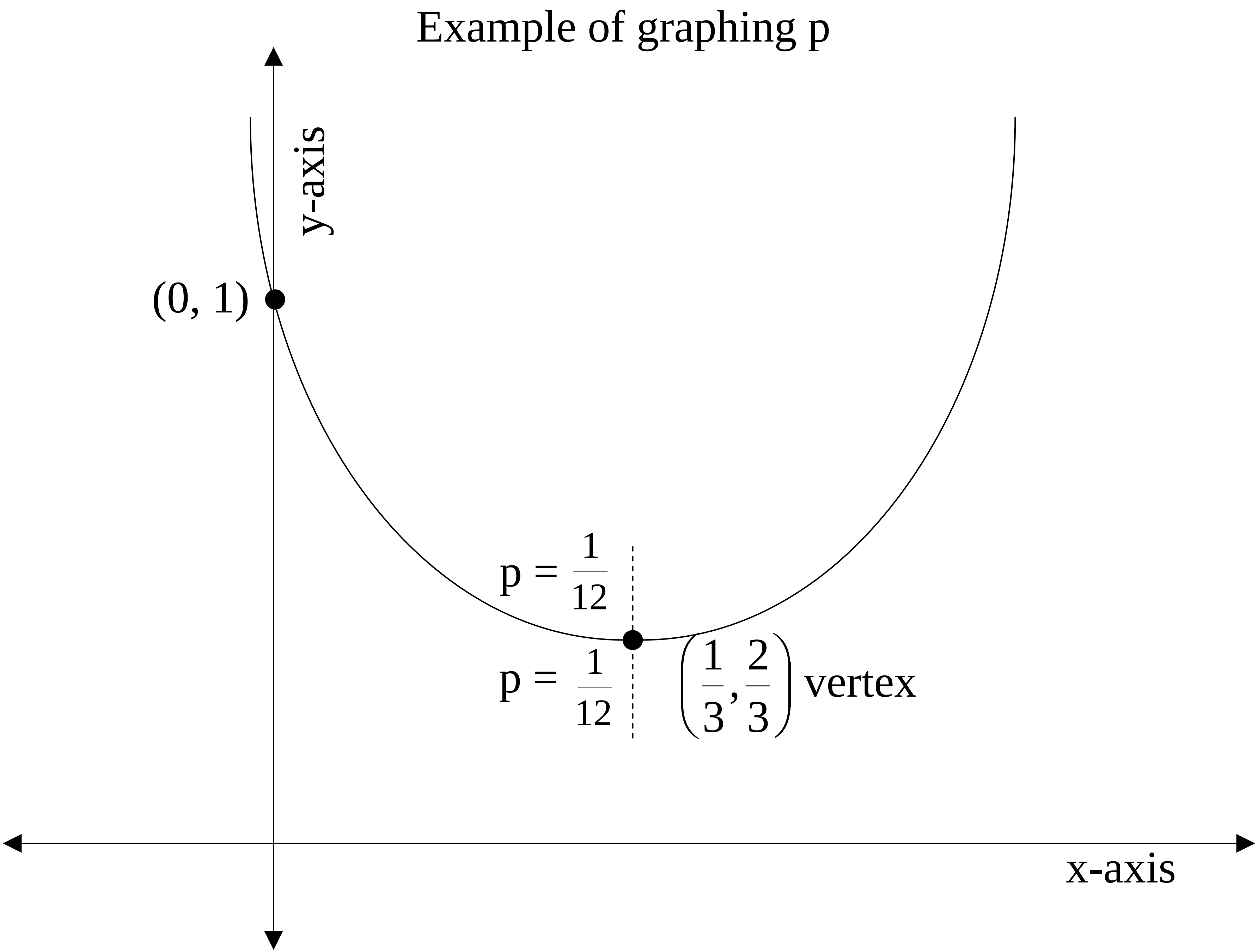
Example of solving for the value of p

$$(x-h)^2 = 4p(y-k)$$

$$\left(x-\frac{1}{3}\right)^2 = \frac{1}{3}\left(y-\frac{2}{3}\right)$$

$$4p = \frac{1}{3}$$

$$p = \frac{1}{12}$$


Example of graphing p
y-axis
(0, 1)
p = 1/12
p = 1/12
(1/3, 2/3) vertex
x-axis

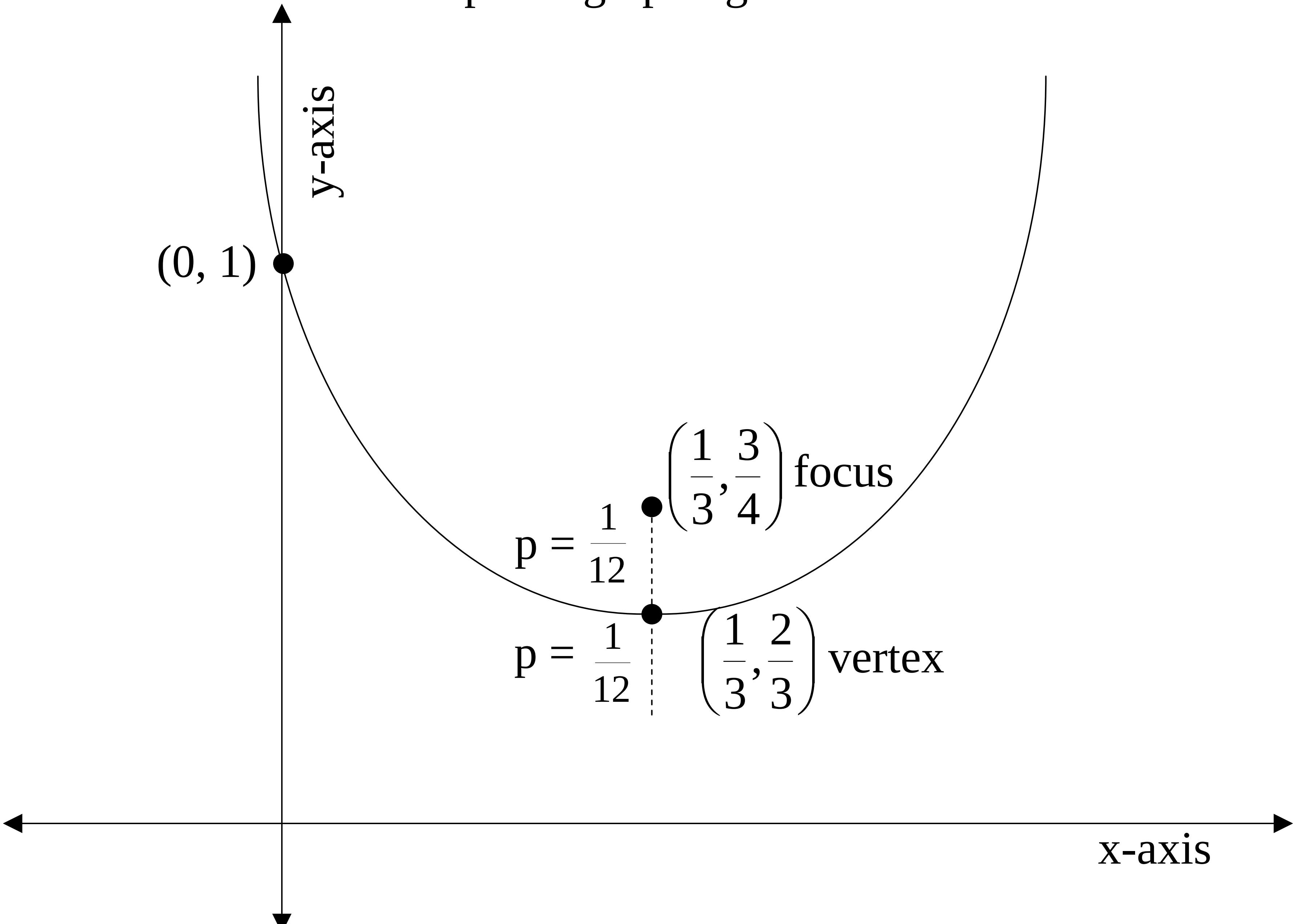
Example of locating the focus

$$\text{vertex}\left(\frac{1}{3},\frac{2}{3}\right) \quad p=\frac{1}{12}$$

$$\text{focus}\left(\frac{1}{3},\frac{2}{3}+\frac{1}{12}\right)$$

$$\text{focus}\left(\frac{1}{3},\frac{3}{4}\right)$$

Example of graphing the focus



Example of locating the directrix & axis of symmetry

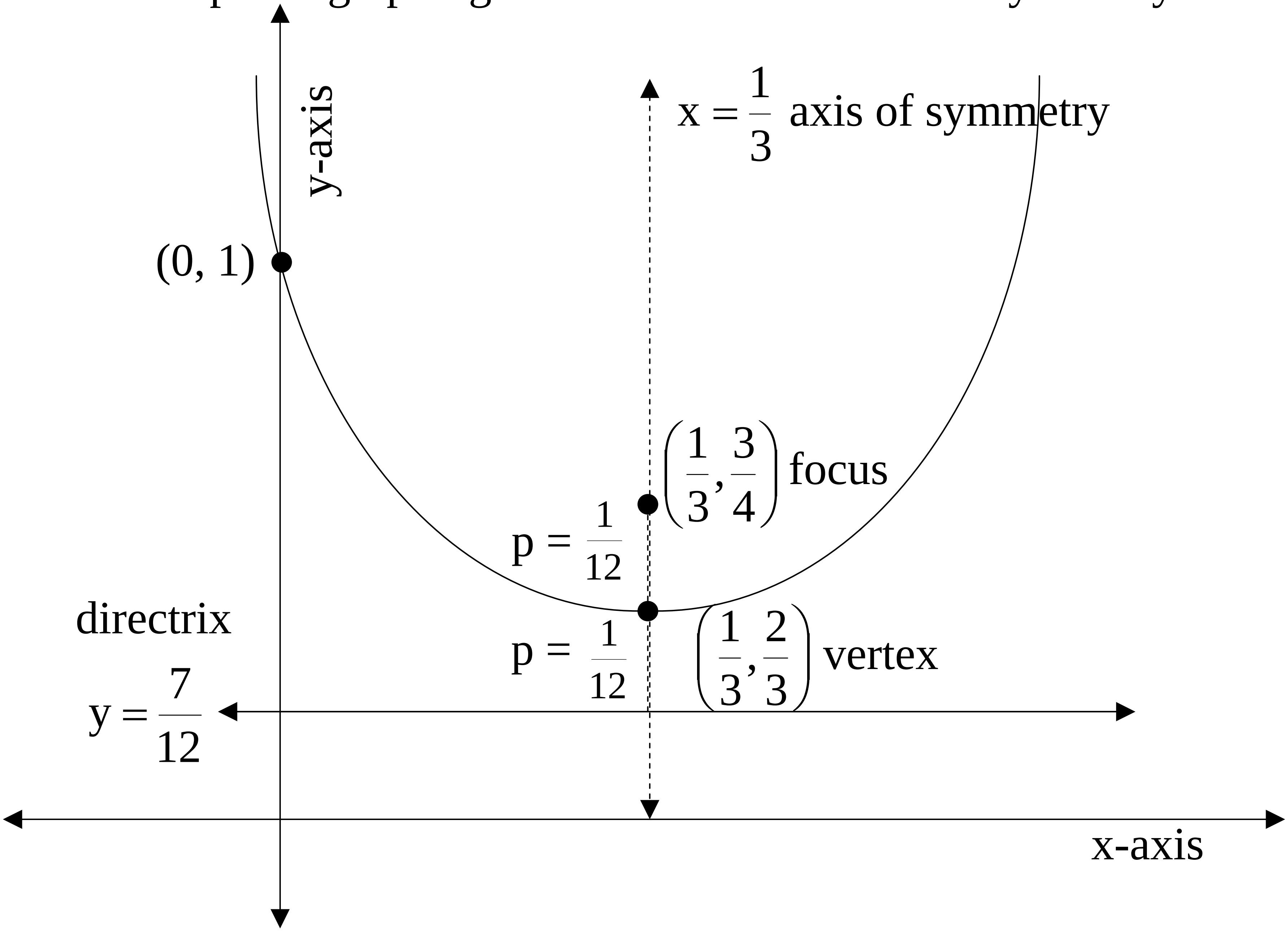
$$\text{vertex}\left(\frac{1}{3},\frac{2}{3}\right) \quad p=\frac{1}{12}$$

$$\text{directrix: horizontal line with the equation } \; y=\frac{2}{3}-\frac{1}{12}$$

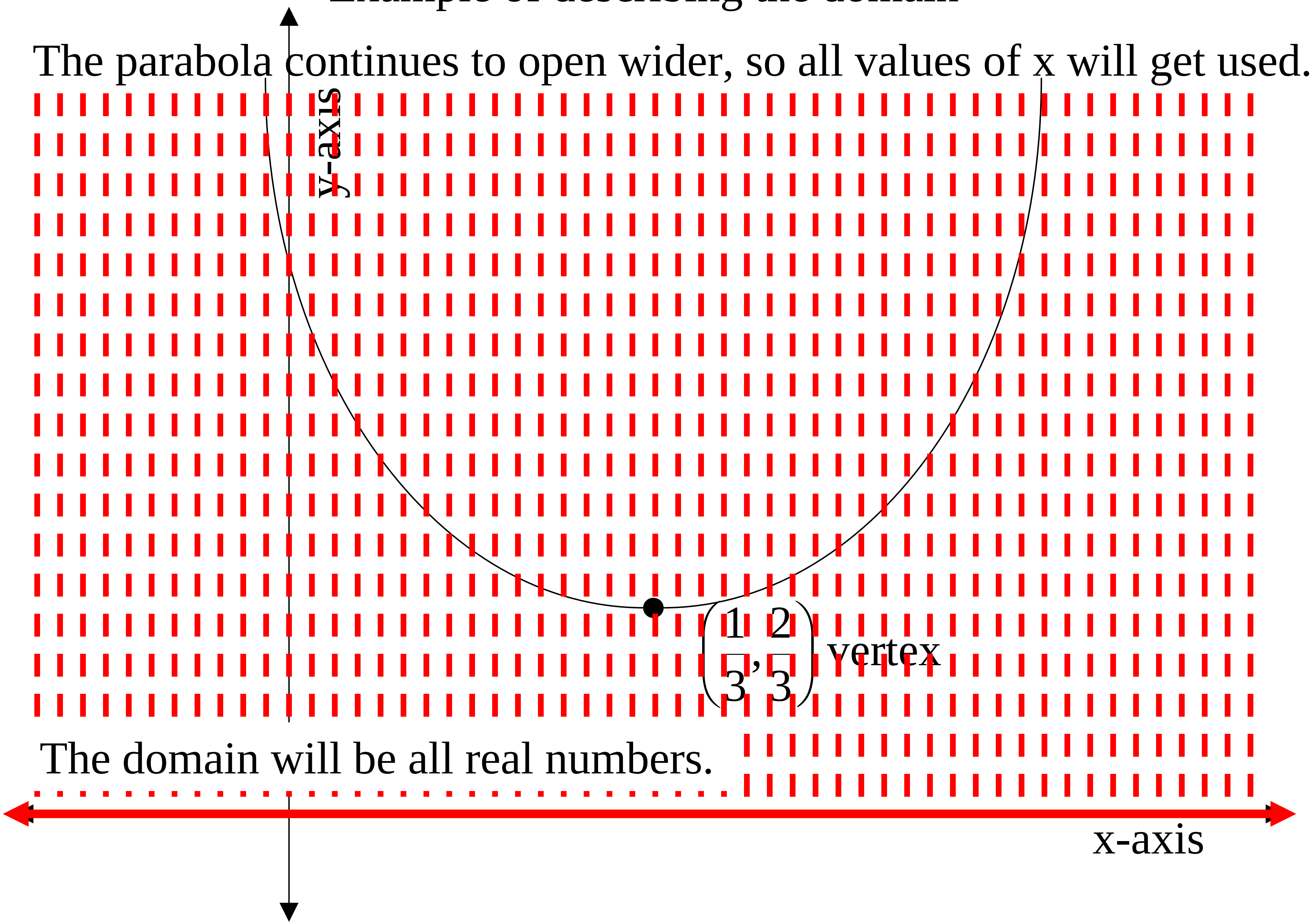
$$y=\frac{7}{12}$$

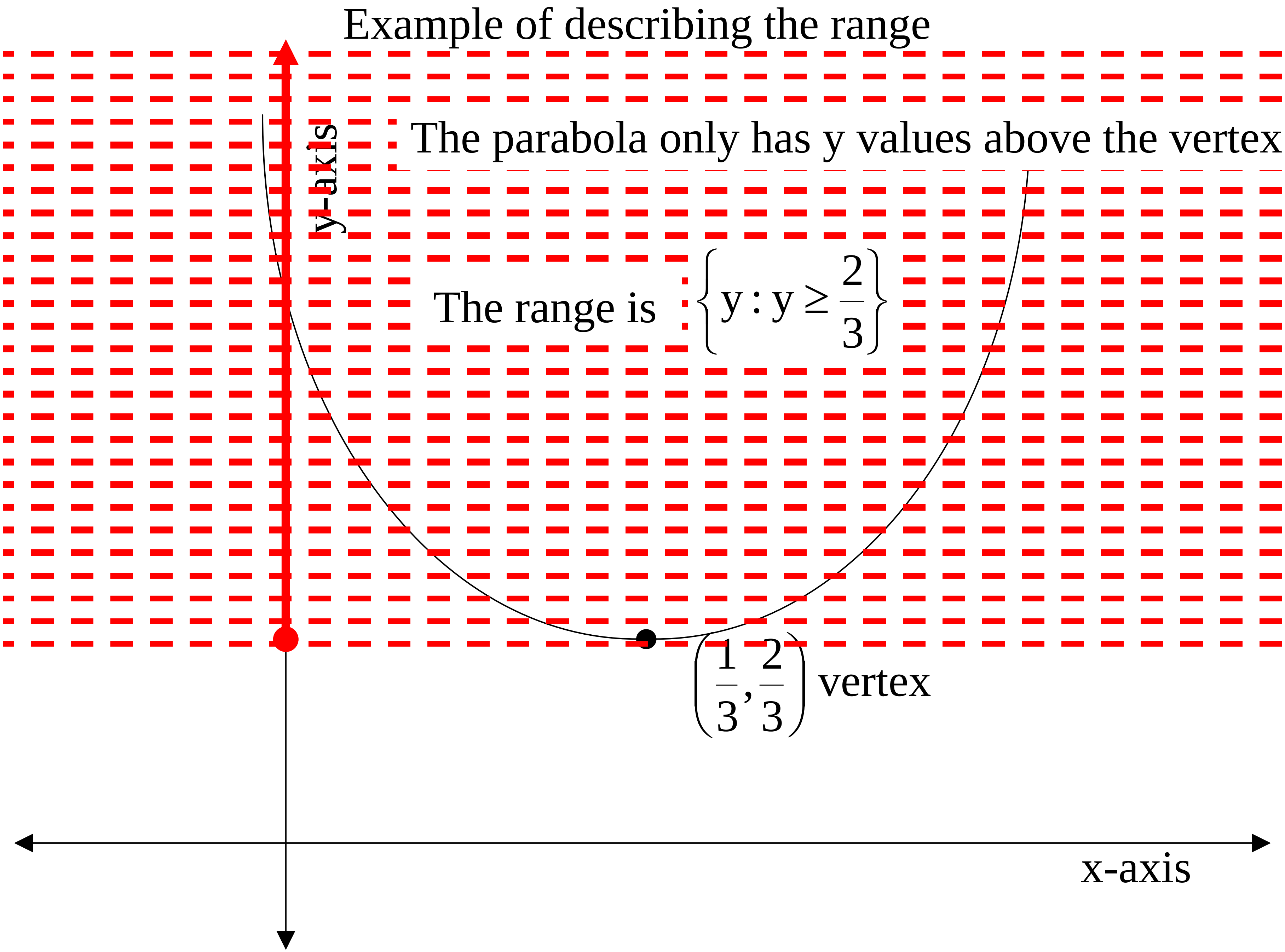
$$\text{axis: vertical line through the vertex with the equation } \; x=\frac{1}{3}$$

Example of graphing the directrix & the axis of symmetry



Example of describing the domain

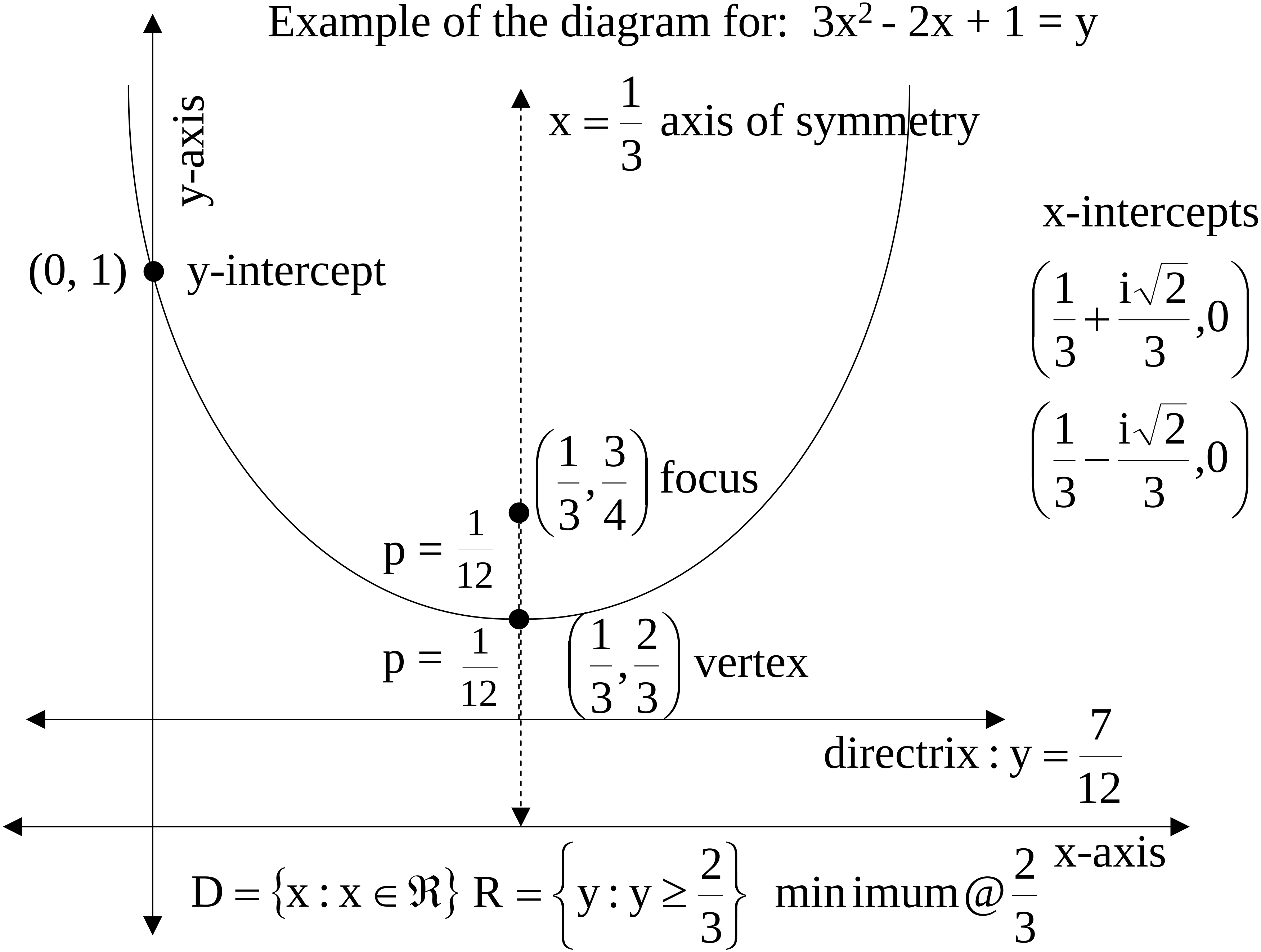



Example of describing the range
y-axis
The parabola only has y values above the vertex
The range is $\left\{ y : y \geq \frac{2}{3} \right\}$
$\left(\frac{1}{3}, \frac{2}{3} \right)$ vertex
x-axis

Example of finding the maximum and minimum values

- The value of an equation is an element of the range.
- A maximum is the largest y value: a minimum is the smallest y value.
- A graph has either a maximum, a minimum or neither.
- This graph has a minimum y value of $\frac{2}{3}$

Example of the diagram for: $3x^2 - 2x + 1 = y$



Do not follow the directions or the formula in the book on problems 2-36. Identify and plot the 11 characteristics you learned yesterday.

Section 7-6

Quadratic Functions

Objectives

- to graph quadratic equations

Characteristics of a Parabola

1. coordinates of x -intercept(s)
2. coordinates of y-intercept(s)
3. coordinates of vertex
4. opens up, down, left or right
5. value of p
6. coordinates of focus
7. equation of the directrix
8. equation of the axis of symmetry
9. description of the Domain
10. description of the Range
11. maximum or minimum point

Locating and Graphing the Parts of Standard Quadratic Form

- Rearrange into standard quadratic form
- Find the x-intercepts by substituting 0 in for y and solving.
- Find the y-intercepts by substituting 0 in for x and solving.
- Find the x coordinate of the vertex by using the formula: $-\frac{b}{2a}$
- Find the y coordinate of the vertex by substituting in the x coordinate and solving for y.
- Identify the sigh of the quadratic coefficient (a) and determine the direction of the parabola.
- Solve for the value of p by using the formula: $\left|\frac{1}{4a}\right|$
- Locate the focus & directrix by using p
- Locate the axis of symmetry by using the direction and the vertex.
- Identify the domain & range.
- Identify the maximum or minimum by using the sign of (a) and the vertex.

Example of rearranging into Standard Quadratic From

$h(x) = 2(x - 7)(x + 5)$

$h(x) = 2(x^2 - 2x - 35)$

$h(x) = 2x^2 - 4x - 70$

$y = 2x^2 - 4x - 70$

Example of locating the x-intercepts

$h(x) = 2(x - 7)(x + 5)$

$0 = 2(x - 7)(x + 5)$

$x = 7$ or $- 5$

x-intercepts are $(7, 0)$ & $(- 5, 0)$

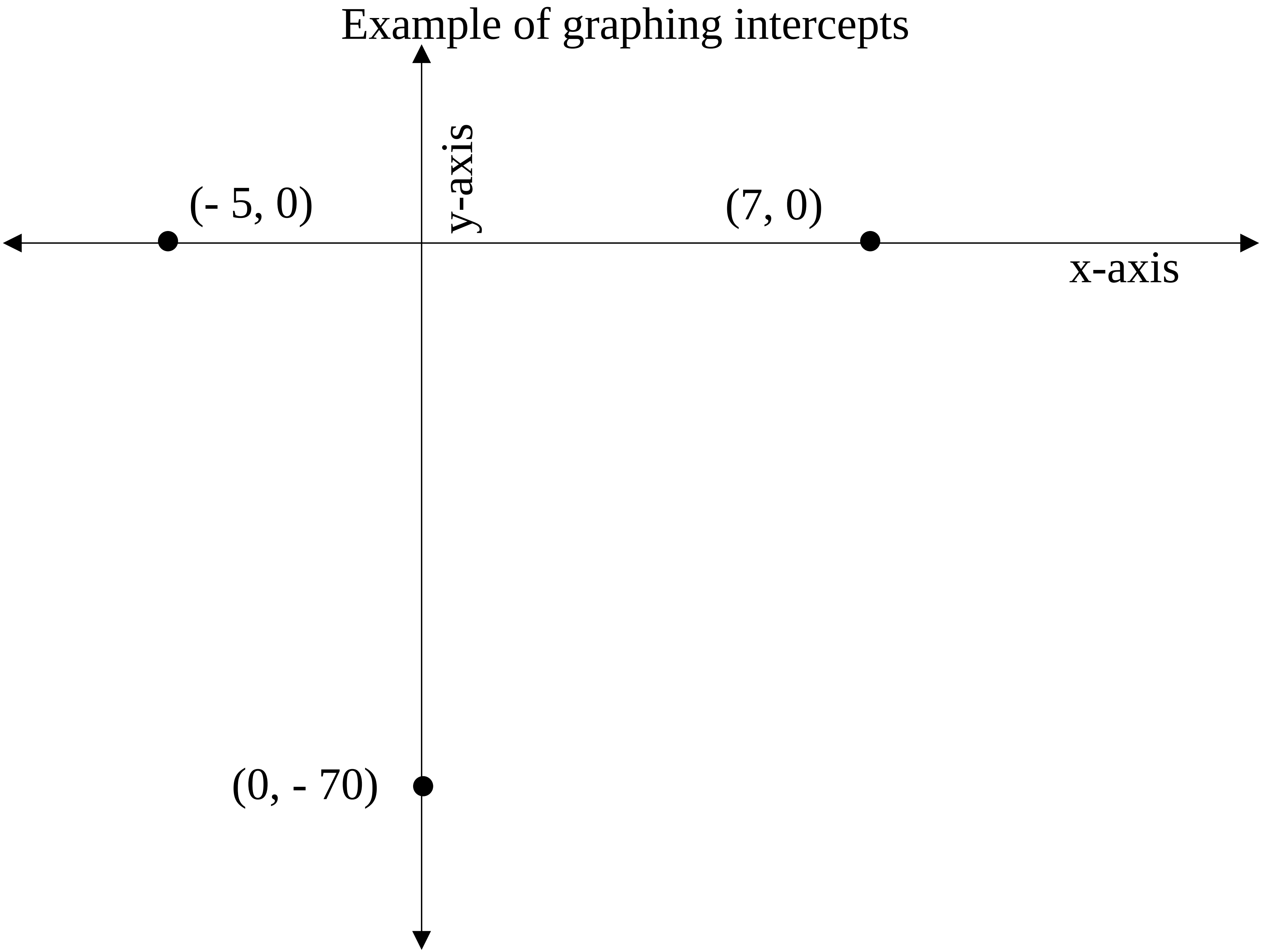
Example of locating the y-intercepts

$y = 2x^2 - 4x - 70$

$y = 2(0)^2 - 4(0) - 70$

$y = - 70$

y-intercept (0, - 70)


Example of graphing intercepts
y-axis
(- 5, 0)
(7, 0)
x-axis
(0, - 70)

Example of finding the x coordinate of the vertex.

$y = 2x^2 - 4x - 70$

$a = 2$ & $b = -4$

$$\text{x coordinate} = -\frac{-4}{2(2)}$$

x coordinate = 1

Example of finding the y coordinate of the vertex

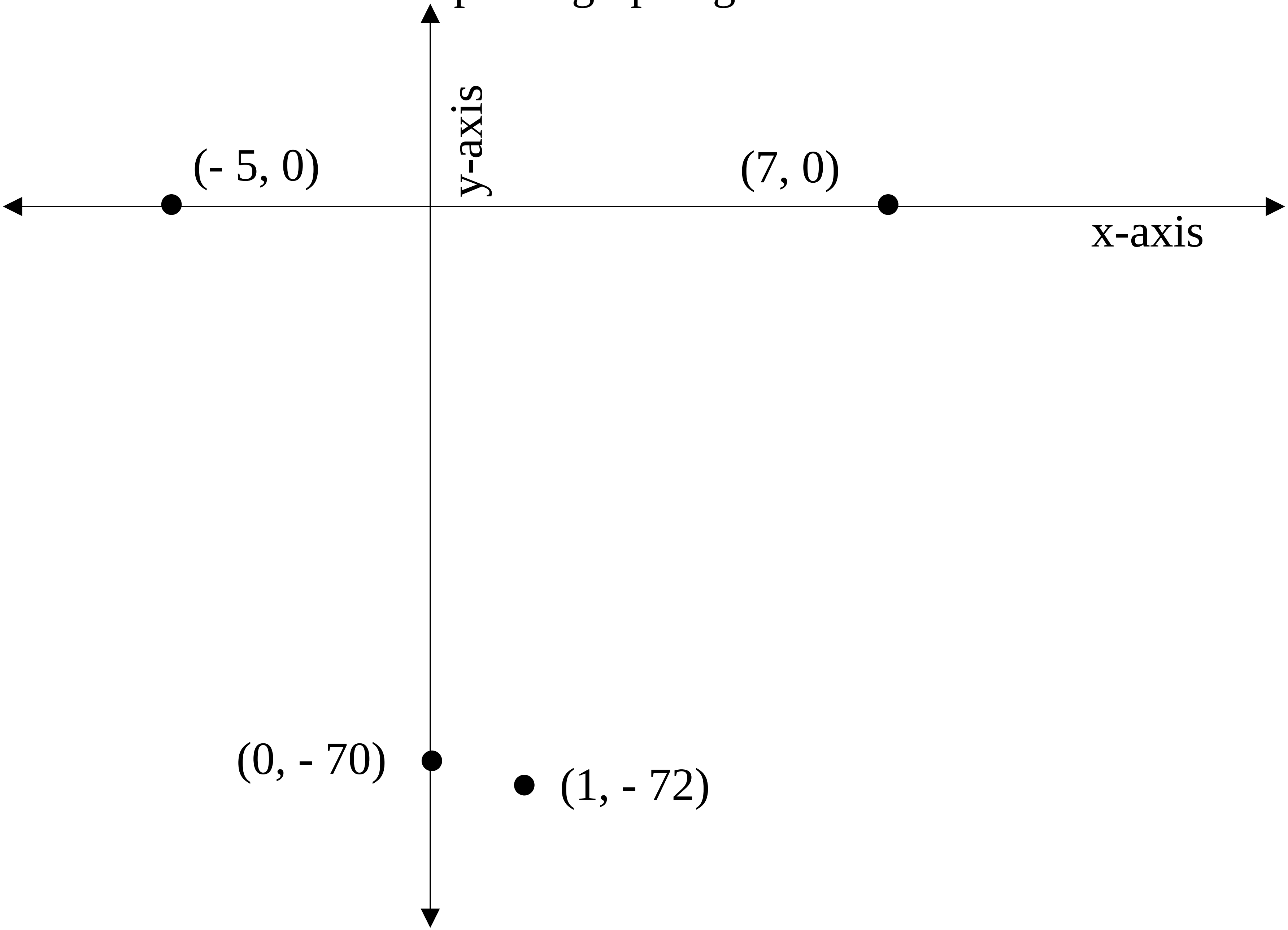
$y = 2(1)^2 - 4(1) - 70$

$y = 2 - 4 - 70$

$y = -72$

vertex $(1, -72)$

Example of graphing the vertex



Example of finding the direction of the parabola

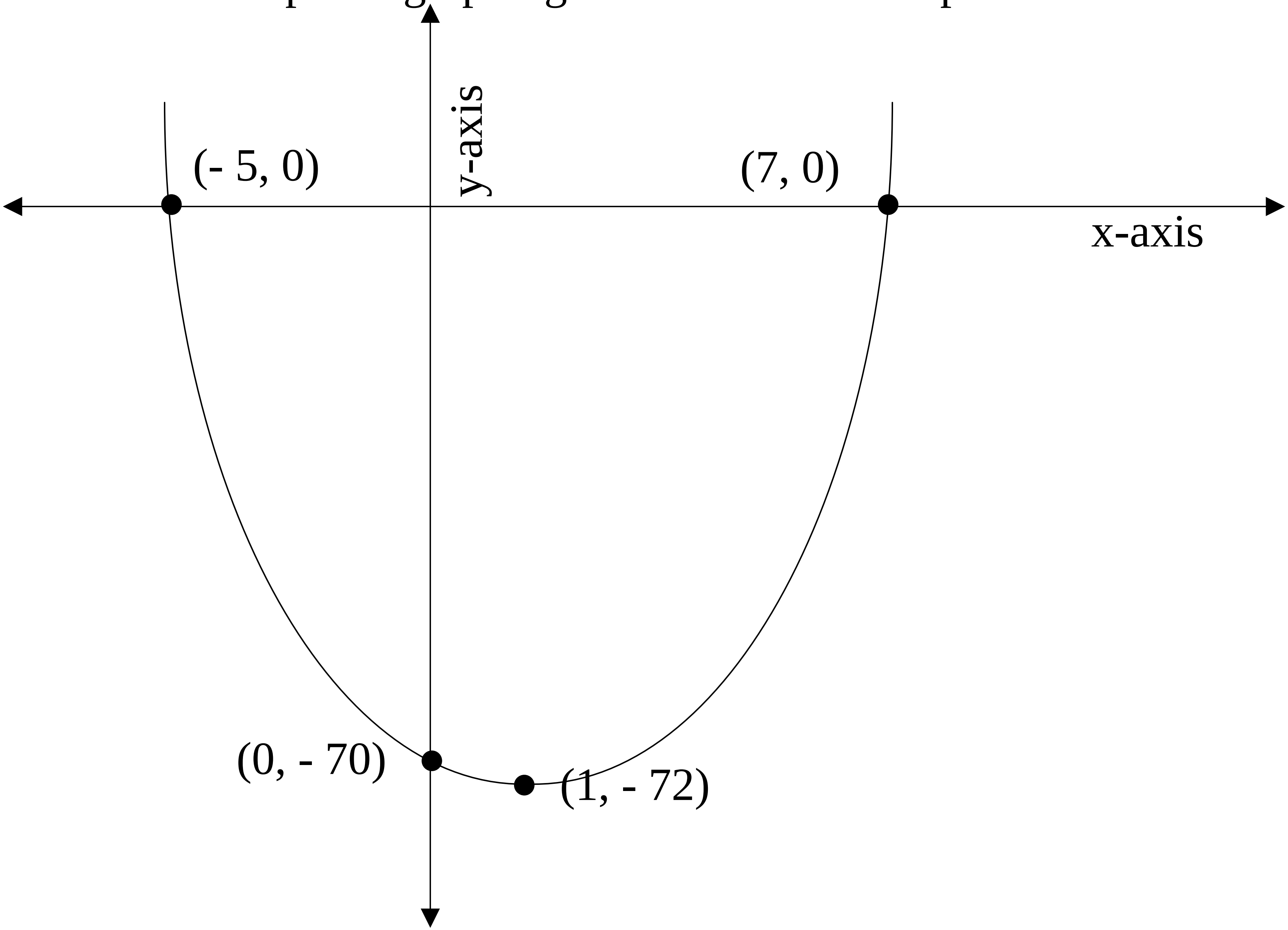
$y = 2x^2 - 4x - 70$

$a = 2$

$2 > 0$

parabola opens up

Example of graphing the direction of the parabola



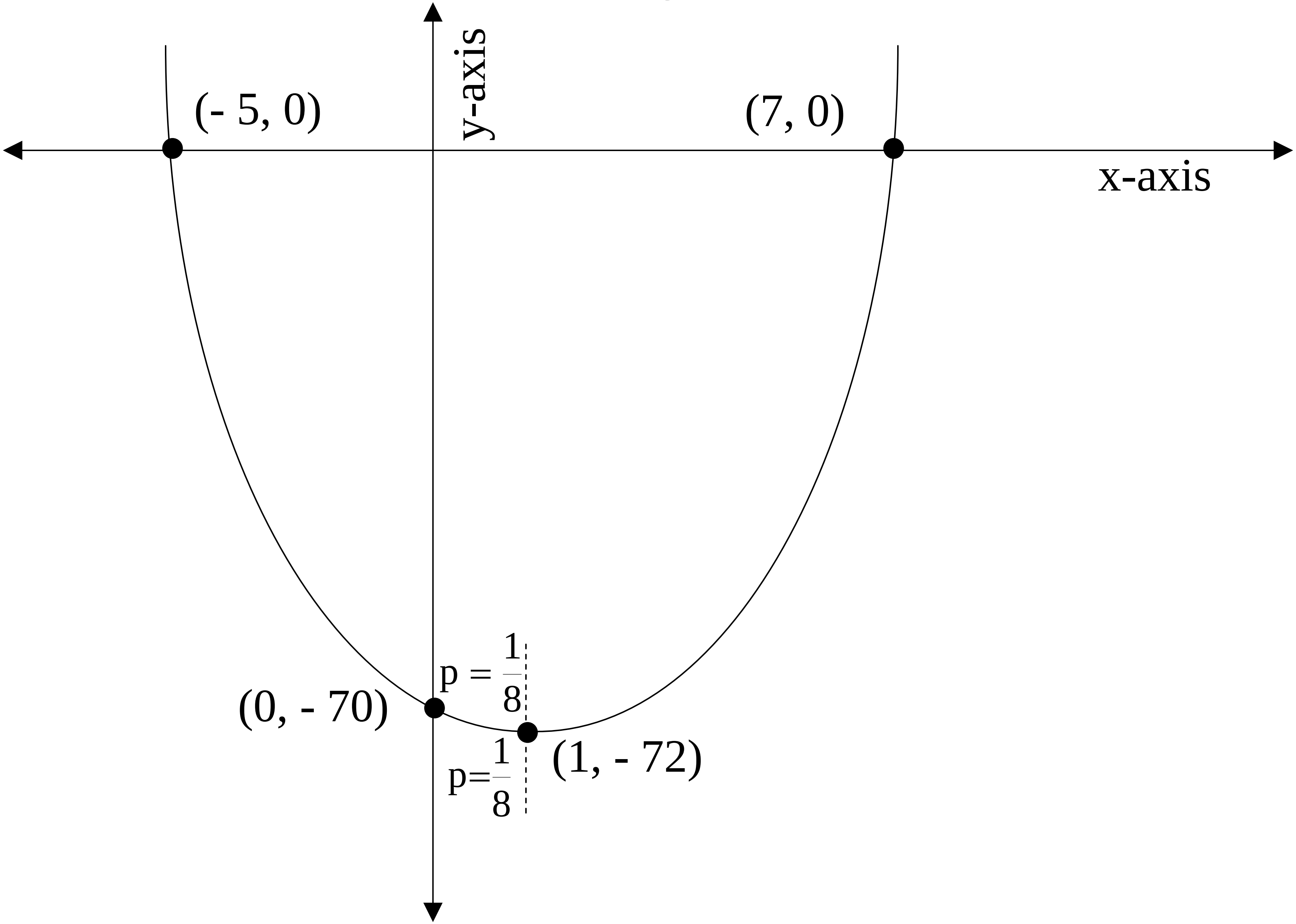
Example of finding the value of p

$y = 2x^2 - 4x - 70$

$a = 2$

$$p = \left|\frac{1}{4(2)}\right| = \left|\frac{1}{8}\right| = \frac{1}{8}$$

Example of plotting the value of (p)

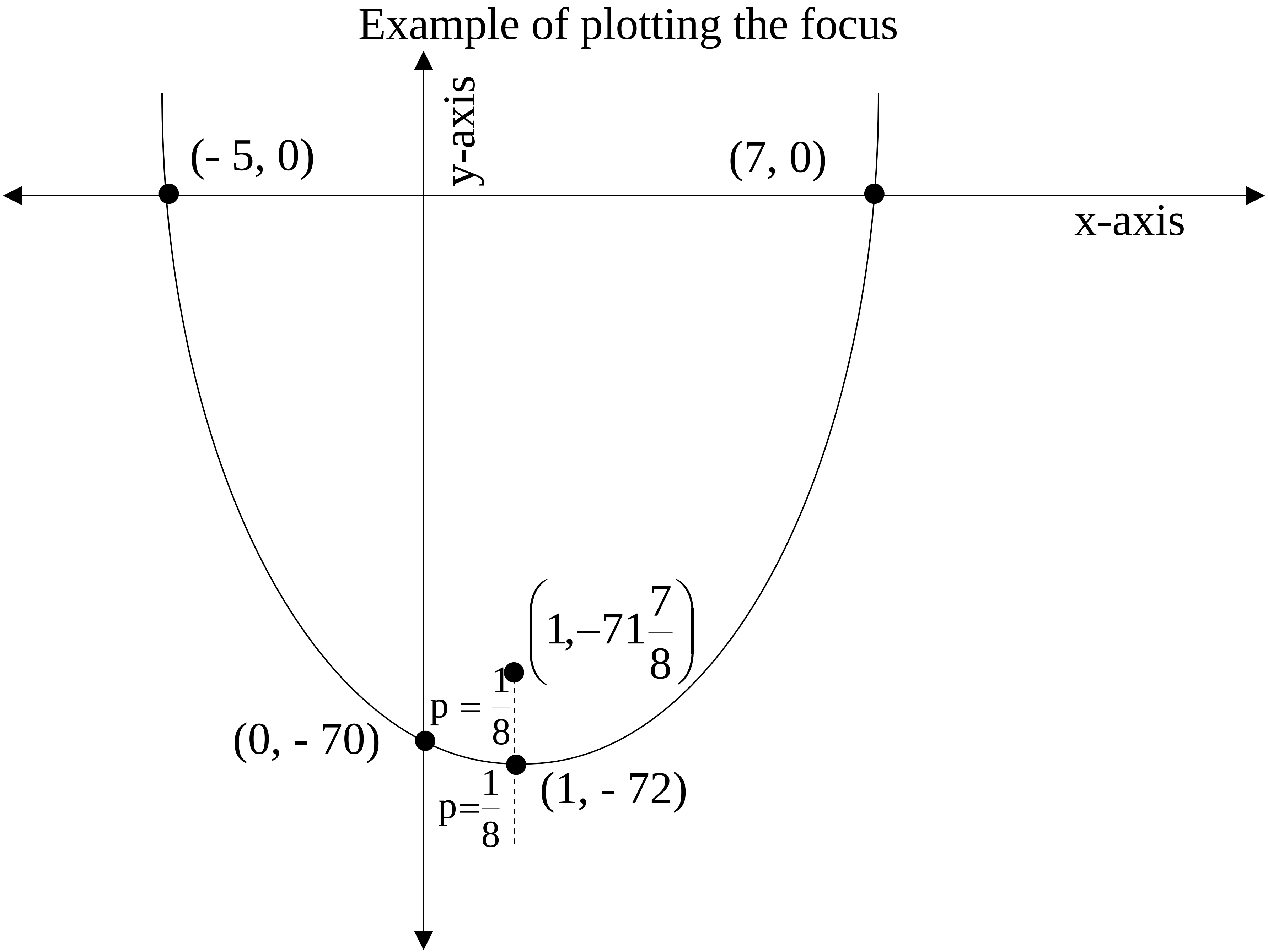


Example of finding the focus

vertex (1, - 72) & parabola opens up

$$\text{focus}: \left(1, -72 + \frac{1}{8}\right)$$

$$\left(1, -71\frac{7}{8}\right)$$



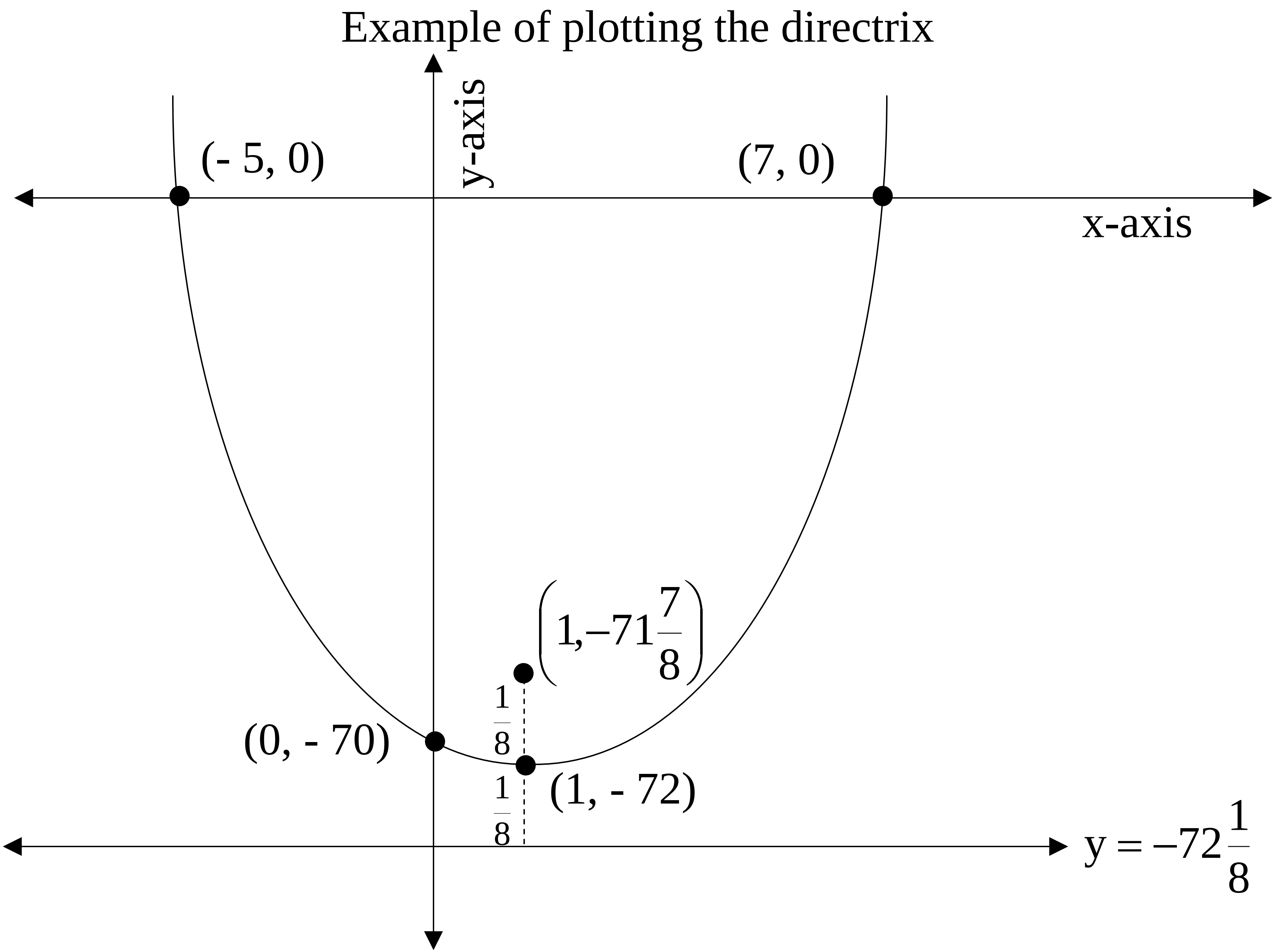
Example of plotting the focus
y-axis
x-axis
(- 5, 0)
(7, 0)
$\left(1, -71\frac{7}{8}\right)$
$p = \frac{1}{8}$
(0, - 70)
(1, - 72)
$p=\frac{1}{8}$

Example of finding the directrix

vertex (1, - 72) & parabola opens up

directrix is a horizontal line (p) units below the vertex

$$y = -72 - \frac{1}{8} = -72\frac{1}{8}$$



Example of plotting the directrix
y-axis
x-axis
(- 5, 0)
(7, 0)
$\left(1, -71\frac{7}{8}\right)$
$\frac{1}{8}$
(0, - 70)
$\frac{1}{8}$
(1, - 72)
$y = -72\frac{1}{8}$

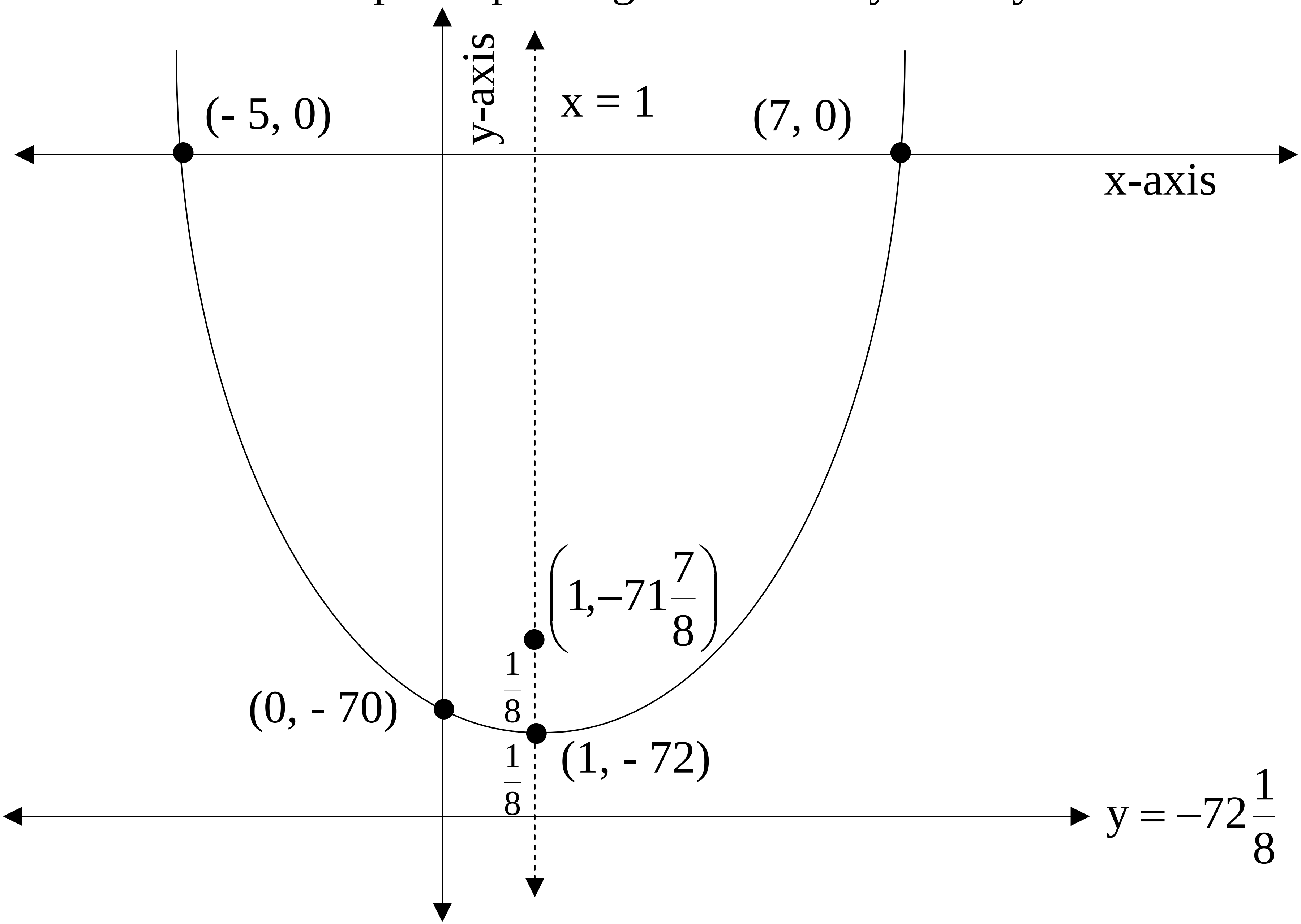
Example of finding the axis of symmetry

vertex (1, - 72) & parabola opens up

the axis of symmetry is a vertical line through the vertex

$x = 1$

Example of plotting the axis of symmetry

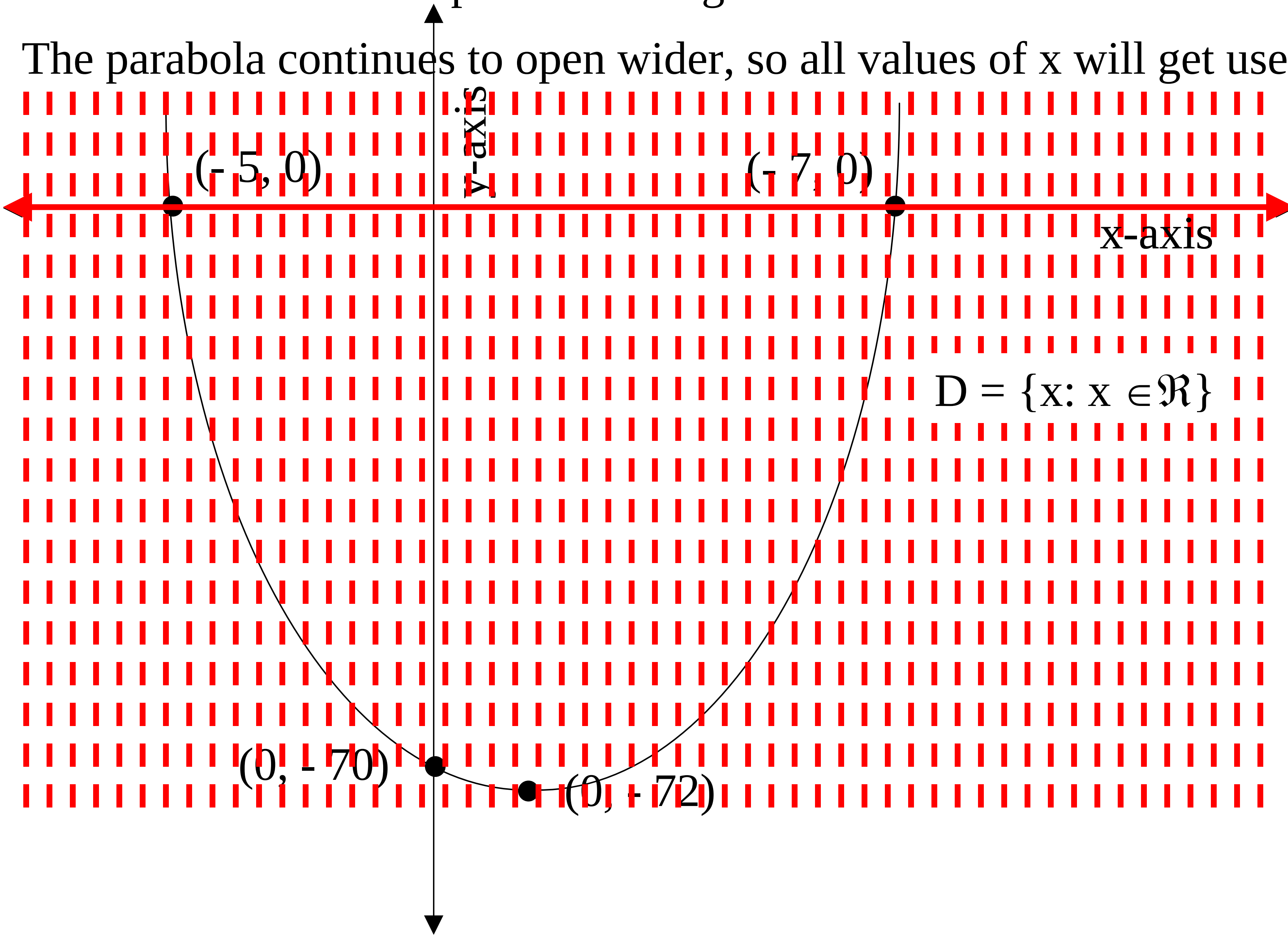


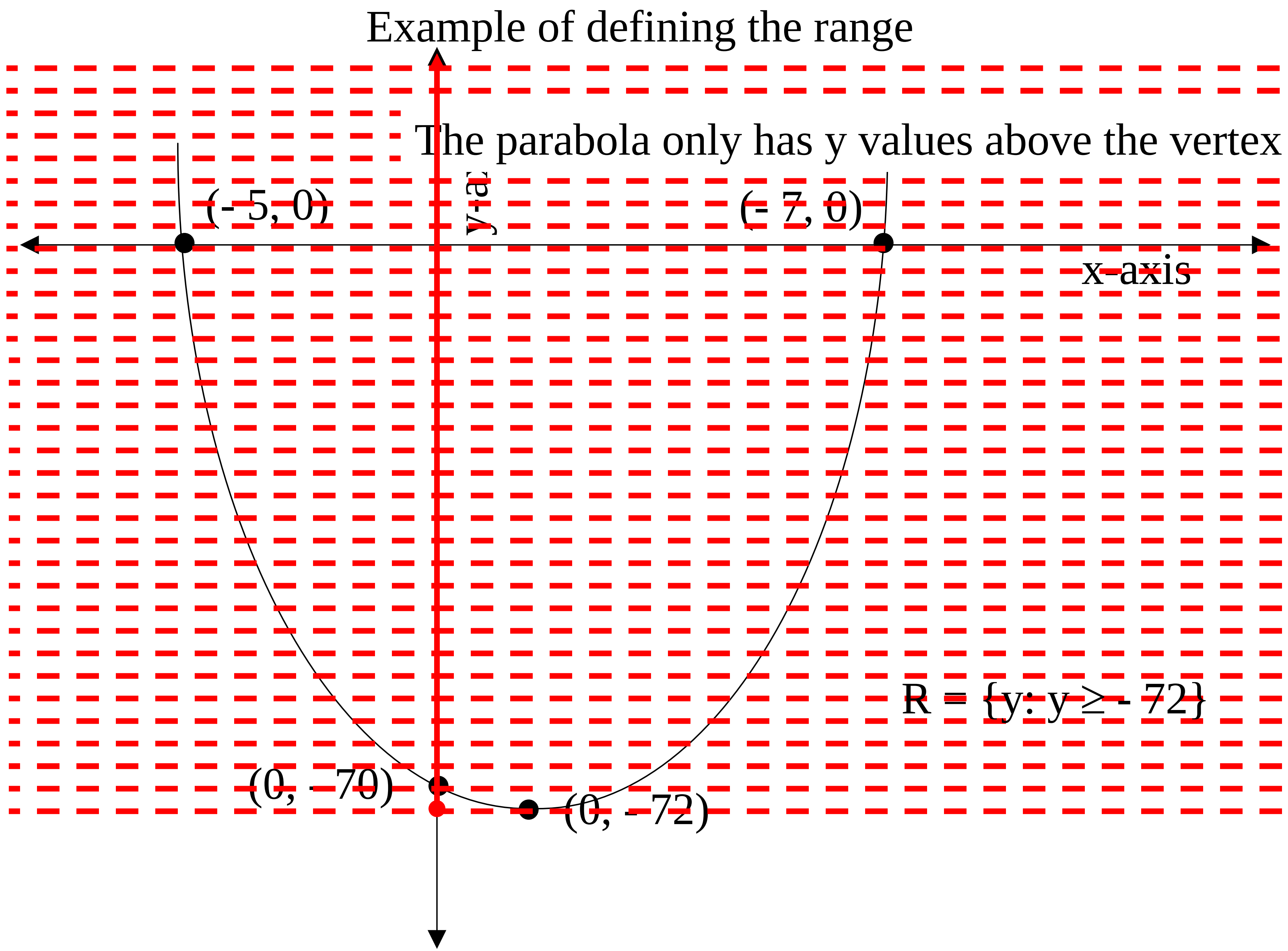
Example of finding the domain, range and max/min

- parabola opens wider; therefore, all values of x are used and D = {x: x $\in \Re$}
- parabola only has points above the vertex; therefore, R = {y: y $\geq$ - 72}
- a = 2 & 2 > 0; therefore, the y coordinate of the vertex is the minimum value @ - 72

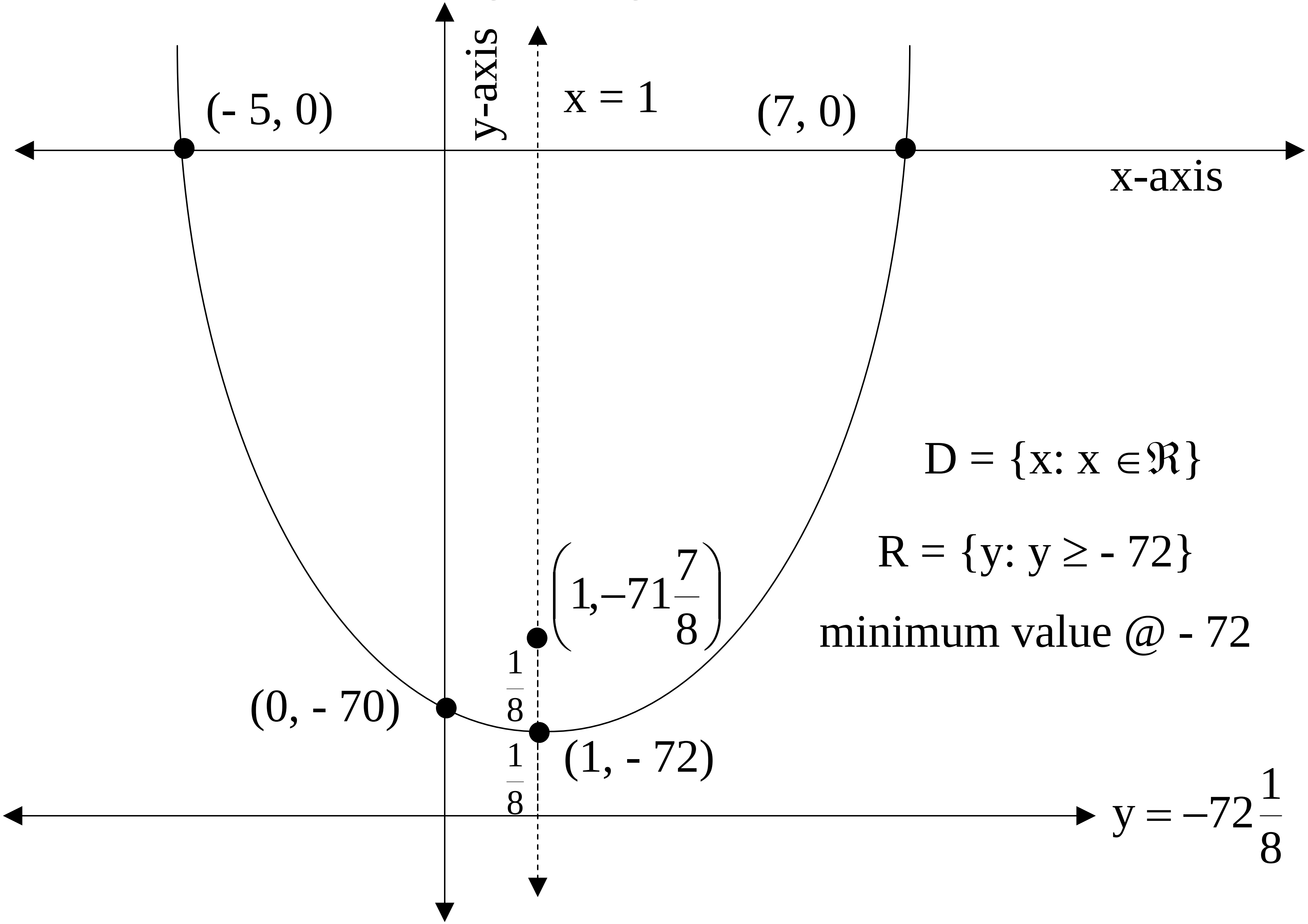
Example of defining the domain

The parabola continues to open wider, so all values of x will get used.




Example of defining the range
The parabola only has y values above the vertex
y-ax
(- 5, 0)
(- 7, 0)
x-axis
R = {y: y ≥ - 72}
(0, - 70)
(0, - 72)

Example of graphing h(x) = 2(x - 7)(x + 5)



Section 7-7

Writing Quadratic Equations and Functions

Objectives

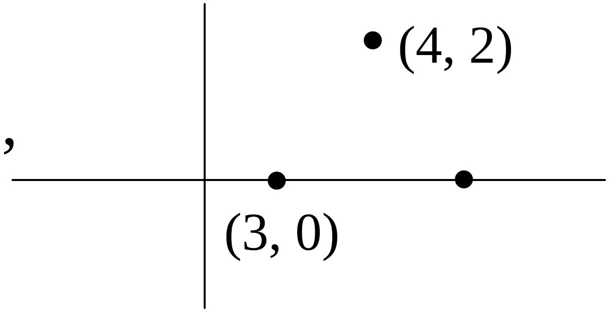
- to write the standard quadratic form of an equation given information about the graph

Writing Standard Parabolic Equations

- plot the information given on a graph to determine the direction of the parabola
- based on the direction, select the appropriate formula
  - opens up: $(x - h)^2 = 4p(y - k)$
  - opens down: $(x - h)^2 = - 4p(y - k)$
  - opens right: $(y - k)^2 = 4p(x - h)$
  - opens left: $(y - k)^2 = - 4p(x - h)$
- substitute the values from your graph into the formula for the corresponding variables
- solve for the missing variable
- the equation is complete when you have the values of (h), (k), and (p)

Example of writing a Standard Parabolic Equation

- Find the standard parabolic equation of a parabola with a vertex (4, 2) and one x intercept 3.
- From a rough graph you can tell that the parabola is opening down; therefore, you would use the equation
  $(x - h)^2 = - 4p(y - k)$



- h = 4, k = 2, x = 3 & y = 0; therefore, you must solve for p
  $(3 - 4)^2 = - 4p(0 - 2)$
  $(- 1)^2 = (- 4)(- 2)p$
  $1 = 8p$
  $p = \frac{1}{8}$
- the equation will be: $(x-4)^2 = -\frac{1}{2}(y-2)$

Writing Standard Quadratic Equations

- If r_1 & r_2 are the roots of a quadratic equation, then that equation can be written as: $a[x^2 - (r_1 + r_2)x + r_1r_2] = y$

$$(r_1 + r_2) = -\frac{b}{a} \qquad r_1r_2 = \frac{c}{a}$$

- Roots, solutions, x-intercepts or zeros are synonyms.
- Identify which variables you have been given and substitute the values into the formula. You must know the values of (a), (r_1) and (r_2) to write the equation.
- Solve for the unknown value.
- Substitute in the values of (a), (r_1) and (r_2) and simplify.

Example of writing a Standard Quadratic Equation

Write the equation of the curve with a y-intercept of 2 and x-intercepts of 1 & 5.

y-intercept of 2 means they are giving me the point (0, 2) to substitute into the values of x & y.

x-intercepts of 1 & 5 means they are giving me the values of r_1 & r_2.

So far we have $a[0^2 - (1 + 5)(0) + (1)(5)] = 2$ when we substitute these values into their proper positions in the formula. We must solve for "a".

$a[0 - 0 + 5] = 2$

$a[5] = 2$

$$a = \frac{2}{5}$$

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Now substitute in only the values of "a", r_1 & r_2 into the proper places in the formula.

$$\frac{2}{5}\left[x^2 - (1+5)x + (1)(5)\right] = y$$

Simplify to get the equation of the curve.

$$\frac{2}{5}\left[x^2 - 6x + 5\right] = y$$

$$\frac{2}{5}x^2 - \frac{12}{5}x + 2 = y$$

Finding Maximum & Minimum Points

- For any standard form quadratic (not parabolic) equation in the form $y = ax^2 + bx + c$ or $f(x) = ax^2 + bx + c$:
  - if $a < 0$, then the parabola opens down and the vertex is the maximum point.
  - if $a > 0$, then the parabola opens up and the vertex is the minimum point.
- The coordinates of the vertex can be written as the point:

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) = \left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$$

- The distance "p" from the vertex to the focus or the distance "p" from the vertex to the directrix is:

$$\left|\frac{1}{4a}\right|$$