Section 9-1

Distance and Midpoint

Objectives

- to solve for the distance between two points
- to find the midpoint between two points
- to find an endpoint given the midpoint and the other endpoint
- to determine the type and area of a triangle given the coordinates of its vertices
- to determine if points are collinear

Distance & Midpoint Formulas

- The distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- The midpoint between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Example for 1-18

- Find the distance between (3, 2) & (5, 6) by using the distance formula

$$d = \sqrt{(5-3)^2 + (6-2)^2}$$

$$d = \sqrt{4+16}$$

$$d = 2\sqrt{5}$$

$$d = \sqrt{(2)^2 + (4)^2}$$

$$d = \sqrt{20}$$

- Find the midpoint between (3, 2) & (5, 6) by using the midpoint formula

$$\left(\frac{3+5}{2}, \frac{2+6}{2}\right)$$

$$(4,4)$$

$$\left(\frac{8}{2}, \frac{8}{2}\right)$$

Example for 19-24

- Use the midpoint formula to find the endpoint Q if the endpoint P(- 4, 0) and the midpoint M(3, 3).

$$3 = \frac{-4 + x}{2} \text{ and } 3 = \frac{0 + y}{2}$$

6 = - 4 + x and 6 = 0 + y

10 = x and 6 = y

Q(10, 6)

Example for 25-28

- A(- 2, 2) B(2, 1) C(1, - 3)
- Use the distance formula to find AB, BC, and AC to determine if it is isosceles.

$$AB=\sqrt{(2+2)^2+(1-2)^2}=\sqrt{17} \quad AC=\sqrt{(1+2)^2+(-3-2)^2}=\sqrt{34}$$

$$BC=\sqrt{(1-2)^2+(-3-1)^2}=\sqrt{17}$$

yes

- Use slope formula to find the slopes of AB, BC and AC to determine if any are perpendicular.

$$\text{m of } AB=-\frac{1}{4} \quad \text{m of } BC=4 \quad \text{m of } AC=-\frac{5}{3}$$

yes

- If any were perpendicular use the area formula for triangles to calculate the area of the triangle.

$$\text{area}=\frac{1}{2}\left(\sqrt{17}\right)\left(\sqrt{17}\right)=\frac{17}{2}$$

Example for 29-32

- Rather than using the distance formula to determine if the points (1, 2) (7, 4) (- 2, 1) are collinear calculate the slopes between the points if they are equal then the points are collinear.

  slope between (1, 2) & (7, 4) is $\frac{1}{3}$

  slope between (1, 2) & (- 2, 1) is $\frac{1}{3}$

  collinear

Section 9-2

Circles

Objectives

- to find the standard form equation of a circle given its center and its radius
- to graph a circle given its center and its radius
- to identify the center and the radius of a circle given its equation
- to graph circular inequalities
- to find the standard form equation of a circle by completing the square
- to write the equation of a circle given information about its graph

Standard Form of a Circle

- A circle with center (h, k) and radius r has an equation:

$$(x-h)^2+(y-k)^2=r^2$$

Example for 1-8

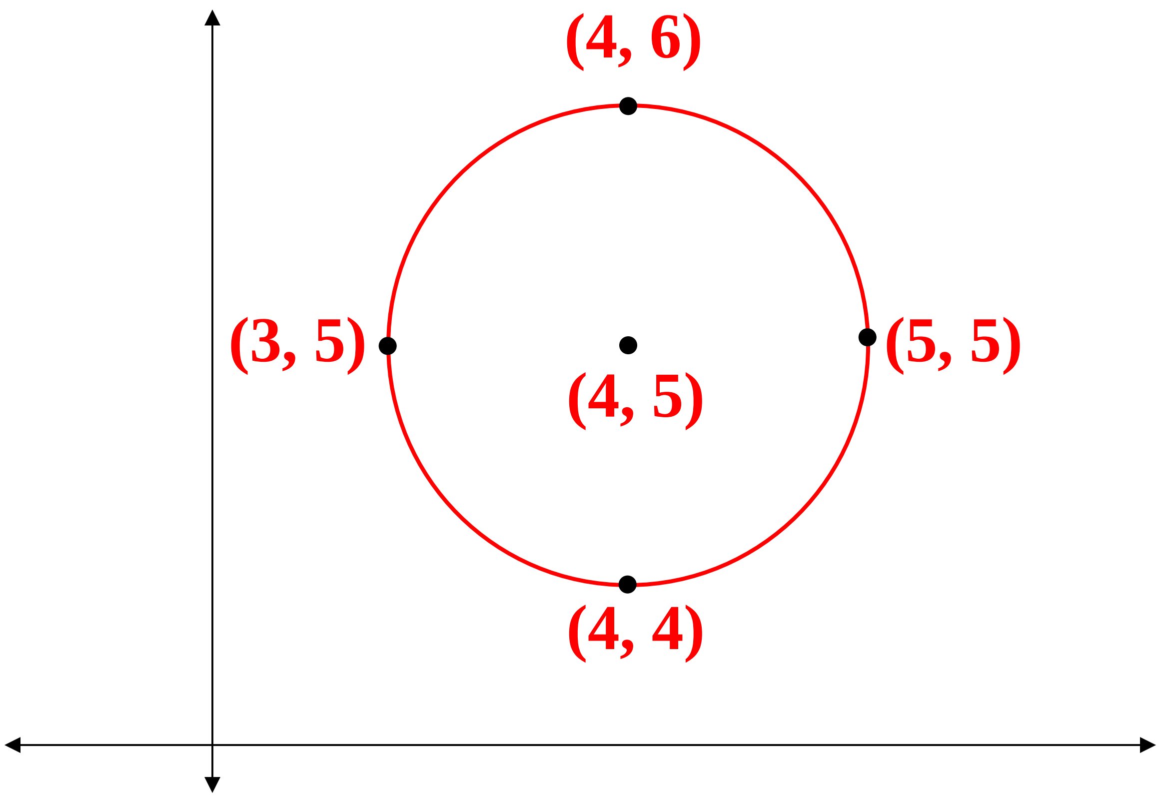
- Substitute the center (2, - 5) in for h and k in the formula and the radius 8 in for r. $(x - h)^2 + (y - k)^2 = r^2$

$$\mathbf{(x - 2)^2 + (y + 5)^2 = 64}$$

Example for 9-14

- Graph the center (h, k) and move a distance r vertically and horizontally from the center to identify four points on the circle. Draw a rough sketch of the circle through these points. $(x - 4)^2 + (y - 5)^2 = 1$

$(h, k) = (4, 5)$
$r^2 = 1 \therefore r = 1$



Example for 15-24

- First rearrange the equation into Standard Circular Form by completing the square. $x^2 + y^2 - 4x + 2y - 4 = 0$

$$\left(x^2 - 4x + 4\right) + \left(y^2 + 2y + 1\right) = 4 + 4 + 1$$

$$(x - 2)^2 + (y + 1)^2 = 9$$

- Use the formula to identify the values of (h, k) for the center and the value of r^2 for the radius.

(h, k) = (2, - 1)

$r^2 = 9 \therefore r = 3$

Example for 25-30

- Repeat the same process you used for problems 9-14 & 15-24 to identify the center and four points on the circle.
- Follow the rules for graphing inequalities when sketching the circle.
- Test the center in the inequality. If the center makes a true statement, then shade inside the circle. If the center makes a false statement, then shade outside the circle.

Example for 31-34

- Complete the square to rearrange the equation into Standard Circular Form.

$4x^2 + 4y^2 - 16x - 24y + 36 = 0$

$x^2 + y^2 - 4x - 6y + 9 = 0$

$(x^2 - 4x + 4) + (y^2 - 6y + 9) = - 9 + 4 + 9$

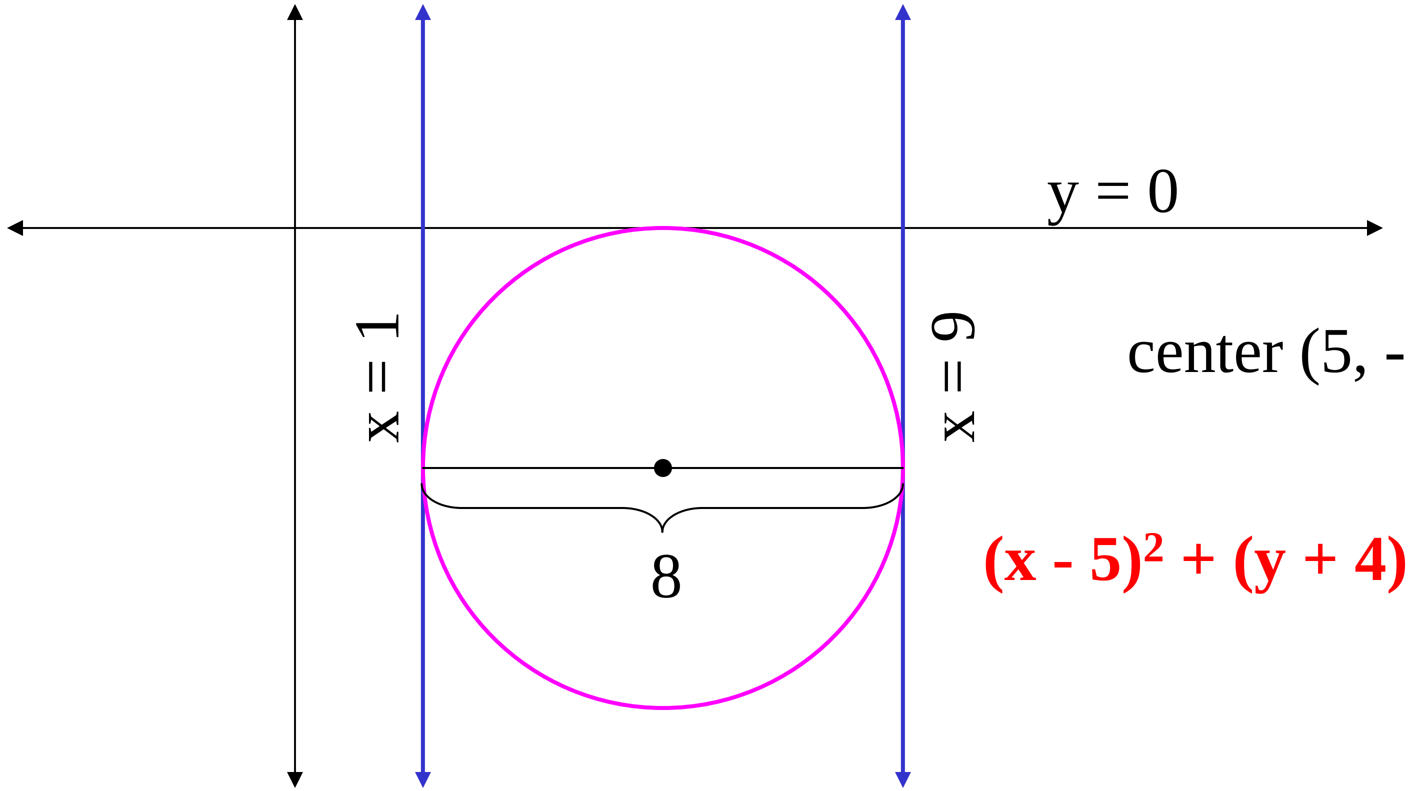
$(x - 2)^2 + (y - 3)^2 = 4$

center (2, 3) r = 2

Example for 35-42

- Every question is giving information about the coordinates of the center and the length of the radius.
- Identify these values and then substitute them into the formula for h, k and r.

Center in quadrant IV; tangent to lines x = 1, x = 9 and y = 0



center (5, - 4) r = 4

$$(x - 5)^2 + (y + 4)^2 = 16$$

Section 9-3

Parabolas

Graph all 11 characteristics

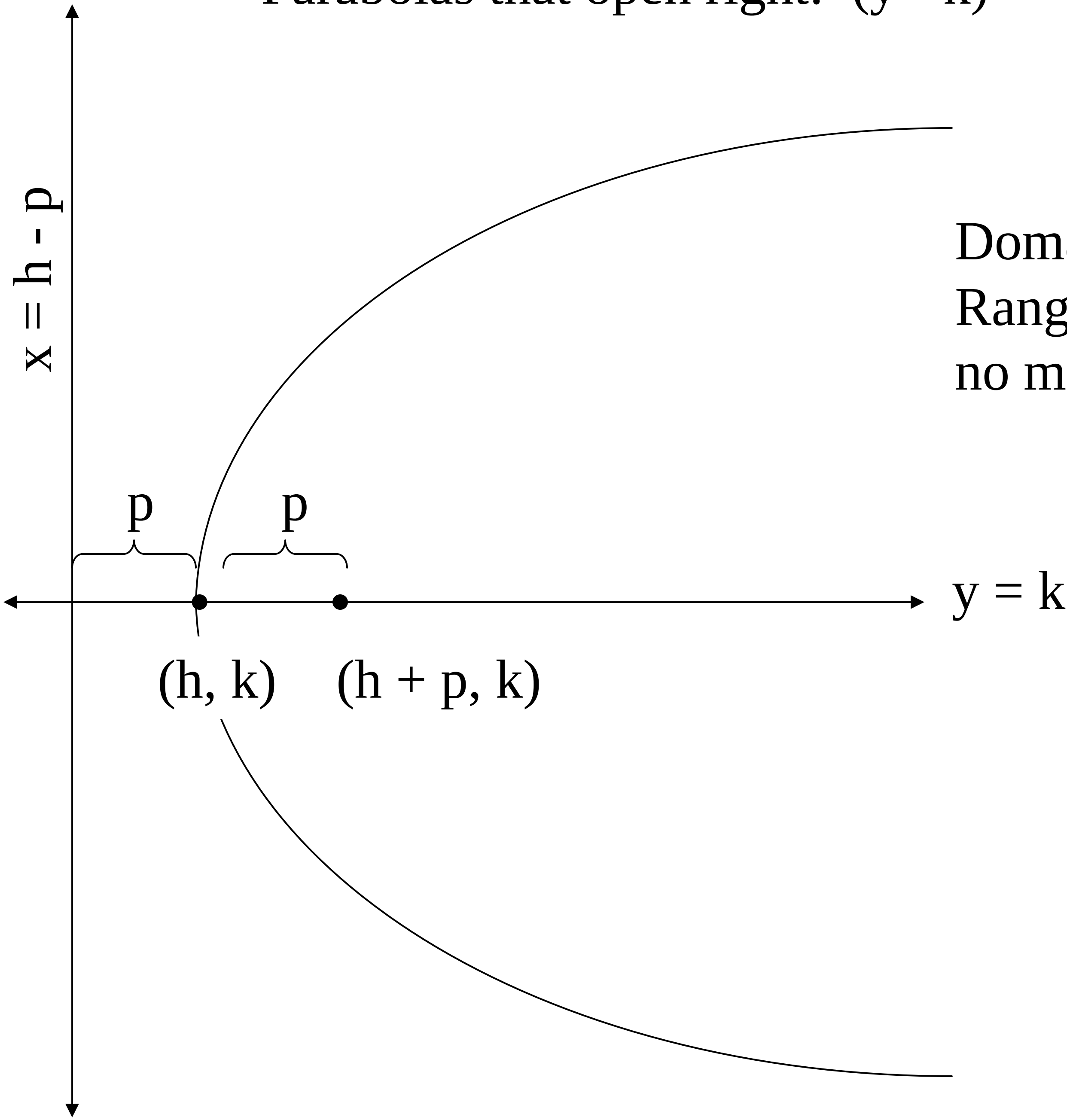
Objectives

- to identify either the focus, vertex or directrix of a parabola when given the other two
- to write and graph the equation of a parabola if given information from the graph
- to graph a parabola
- to graph parabolic inequalities

11 Characteristics of a Parabola

1. x-intercepts
2. y-intercepts
3. vertex
4. direction
5. p
6. focus
7. directrix
8. axis of symmetry
9. domain
10. range
11. maximum/minimum

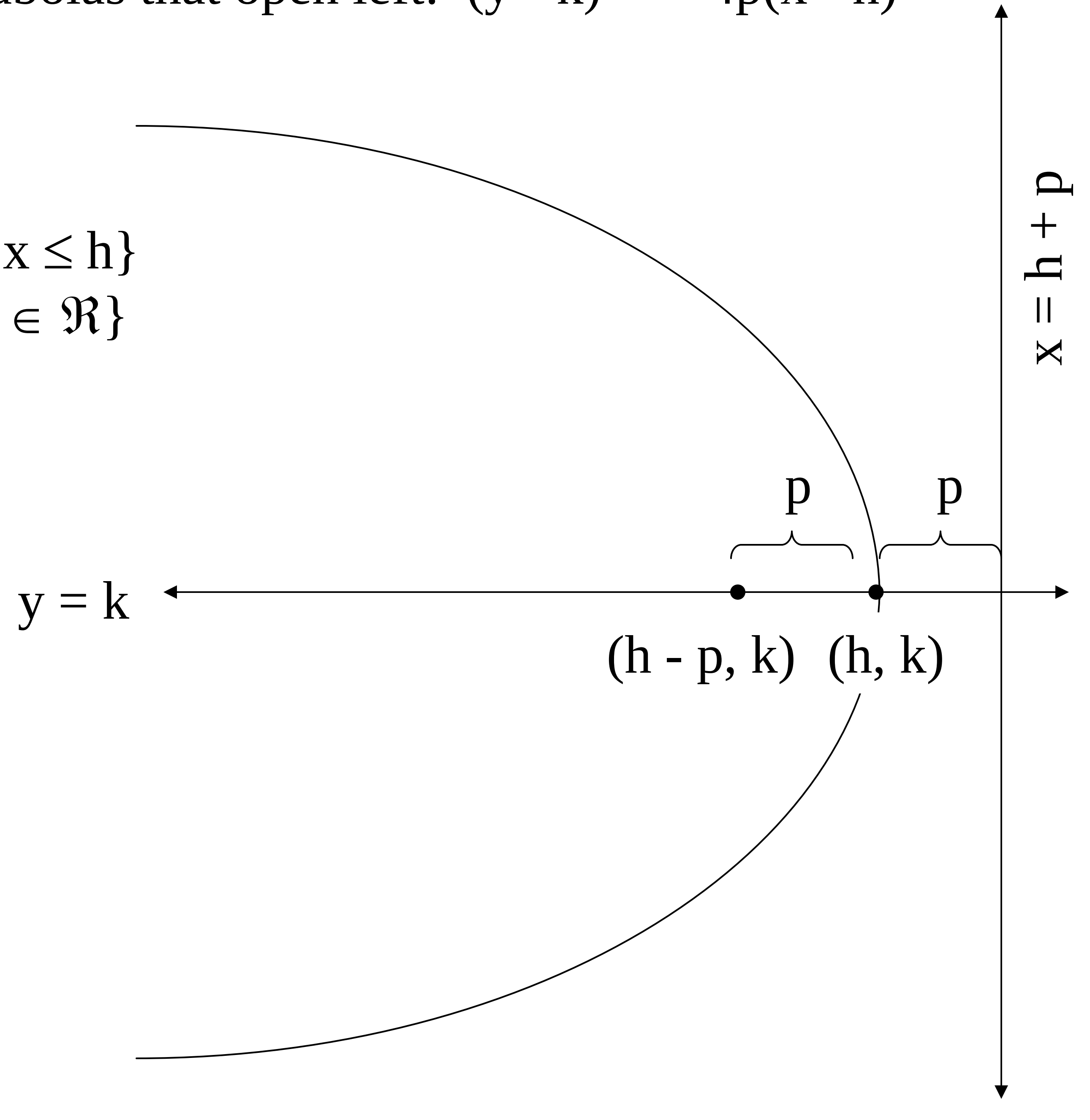
Parabolas that open right: $(y - k)^2 = 4p(x - h)$



Domain $\{x: x \geq h\}$
Range $\{y: y \in \Re\}$
no max/min

Parabolas that open left: $(y - k)^2 = -4p(x - h)$

Domain $\{x: x \leq h\}$
Range $\{y: y \in \Re\}$
no max/min



Example for 1-6

- Remember that the vertex is halfway between the focus and the directrix in a parabola and that the focus and the vertex lie on the same line (vertical if the parabola opens up/down; horizontal if the parabola opens left/right)

  V(4, 2) D: y = - 3

The directrix is a horizontal line so the parabola opens up and the focus will be above the vertex.

The distance from the directrix @ y = - 3 to the vertex @ y = 2 is 5; therefore the focus is 5 units above the vertex.

F(4, 7)

Example for 7-16

- Determine which way the parabola opens based on the relative position of the characteristics identified n the problem. Focus (0, 0) directrix: $y = 4$
  - The directrix is horizontal and above the focus so the parabola opens down.
- Identify the Standard Form Equation for a parabola opening in that direction.
  - The correct equation is: $(x - h)^2 = - 4p (y - k)$
- Identify and substitute in the values of h, k and p.
  - The vertex will have the same x-coordinate as the focus; therefore, $h = 0$.
  - The value of p is half the distance between the focus and the directrix; therefore, $p = 2$ which makes the y-coordinate of the vertex 2.
  - The correct equation is: **$(x - 0)^2 = - 8(y - 2)$**

Example for 17-26

- Graph the parabolas using either method you learned in chapter 7. Include all 11 characteristics in your diagram.
- If y is the variable squared in the problem, then the parabola is opening either left or right.
- I suggest that you complete the square to get the problem into Standard Parabolic Form rather than working from a quadratic equation.

Example for 27-30

- Graph the parabolas as you did in the last section. **You only need to identify the vertex, focus and intercepts.**
- Sketch the parabolic inequalities using the same rules as you did for linear inequalities.
- Test the focus in the inequality. If it makes a true statement, then shade inside the parabola. If it makes a false statement, then shade outside the parabola.

Section 9-4

Ellipses

Identify & graph all 11 characteristics

Objectives

- to identify and graph the eleven characteristics of an ellipse
- to write the standard form equation of an ellipse
- to graph elliptical inequalities

Standard Form for an Ellipse

- An ellipse is the set of all points in a plane such that the sum of the distances from any point P to two fixed points (foci) is a given constant (focal sum).
- Standard Elliptical Form: center (h, k); semi-major axis = a; semi-minor axis = b; center to focus = c = $\sqrt{a^2 - b^2}$
  - major axis is horizontal

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

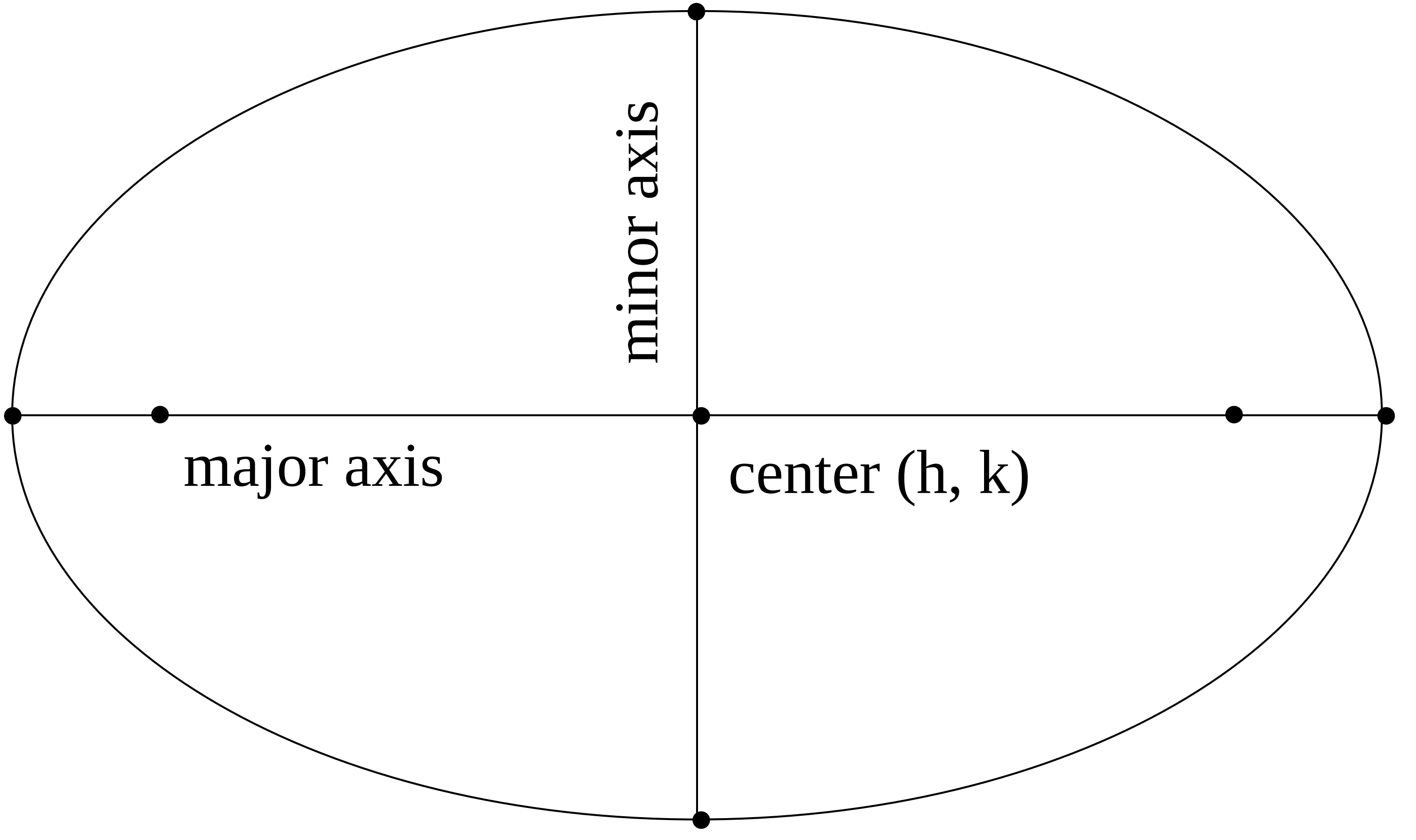
  - major axis is vertical

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

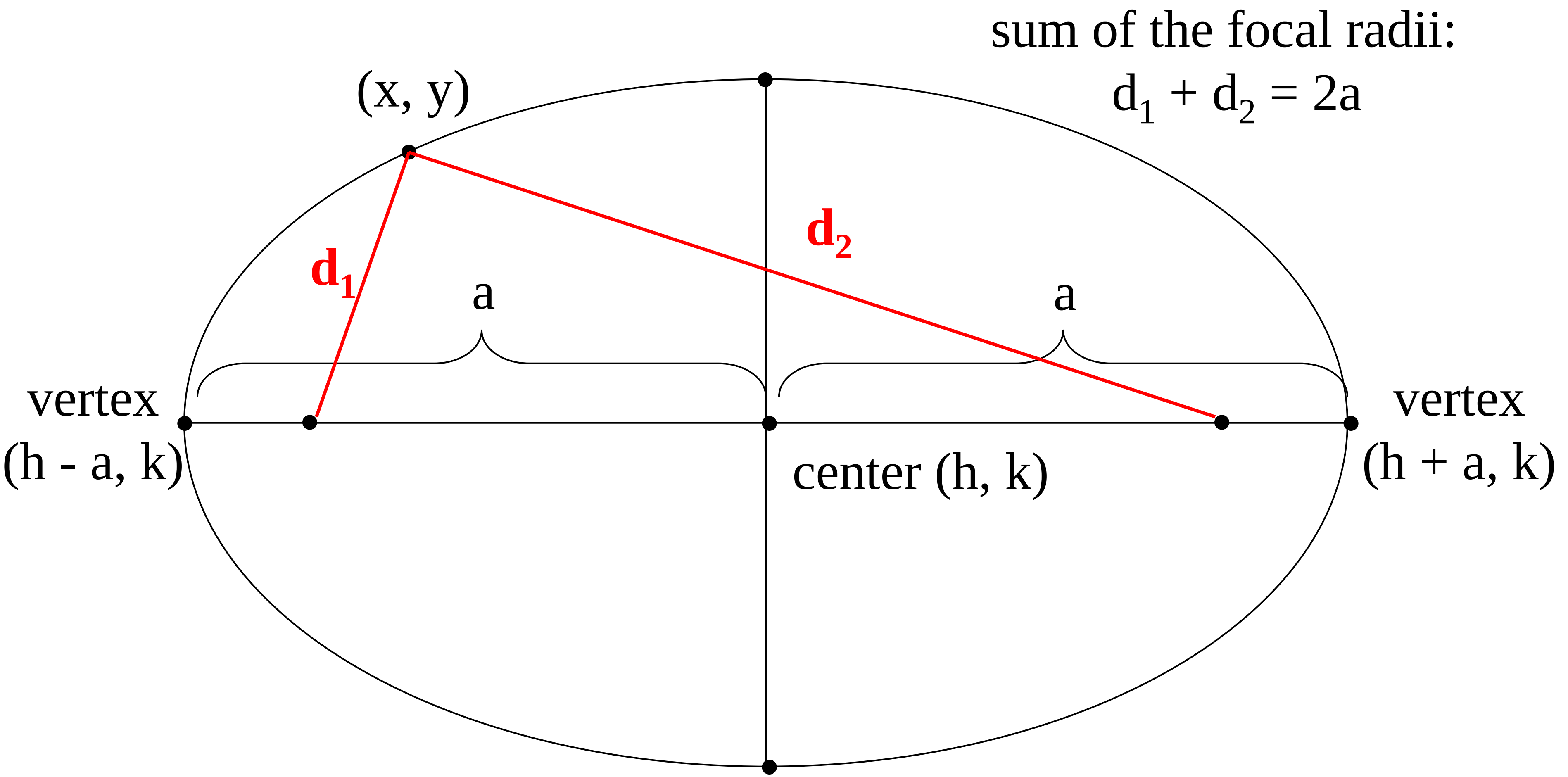
11 Characteristics of an Ellipse

1. value of a: distance from the center to the vertices
2. value of b: distance from the center to the endpoints of the minor axis
3. value of c: distance from the center to the foci
4. center: (h, k)
5. direction: horizontal/vertical major axis
6. coordinates of two vertices
7. coordinates of the endpoints of the minor axis
8. coordinates of the two foci
9. sum of the focal radii
10. domain
11. range

horizontal major axis

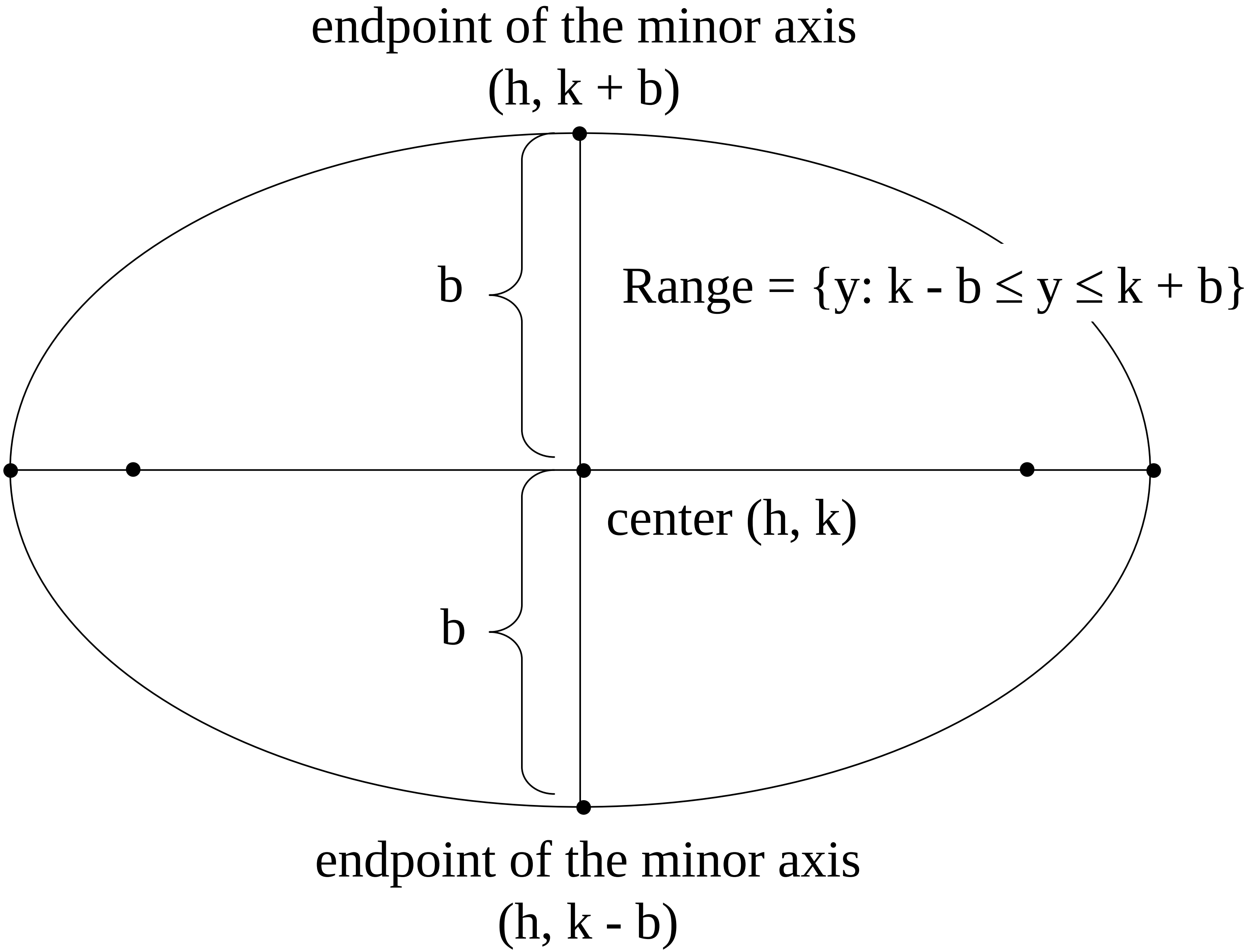


horizontal major axis

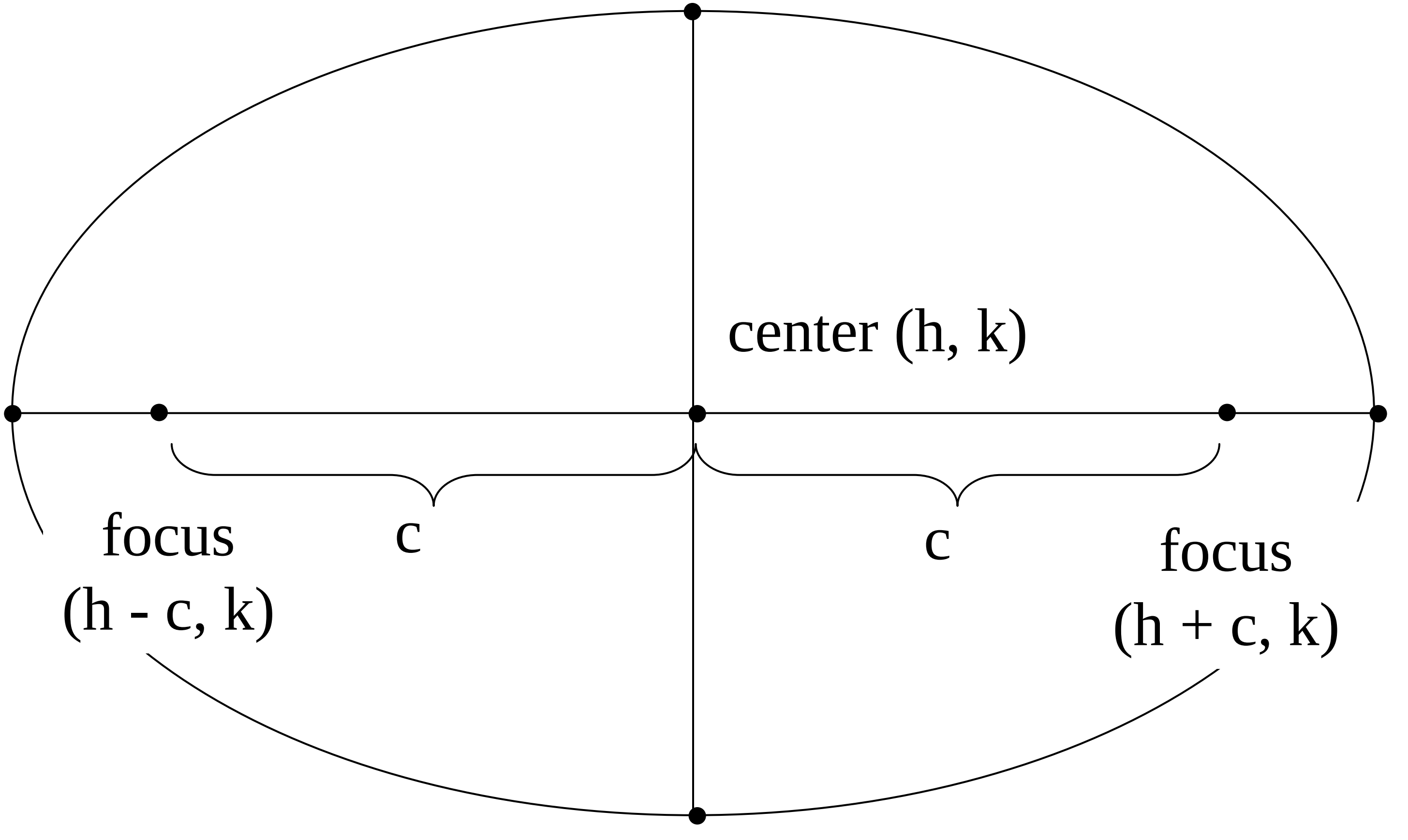


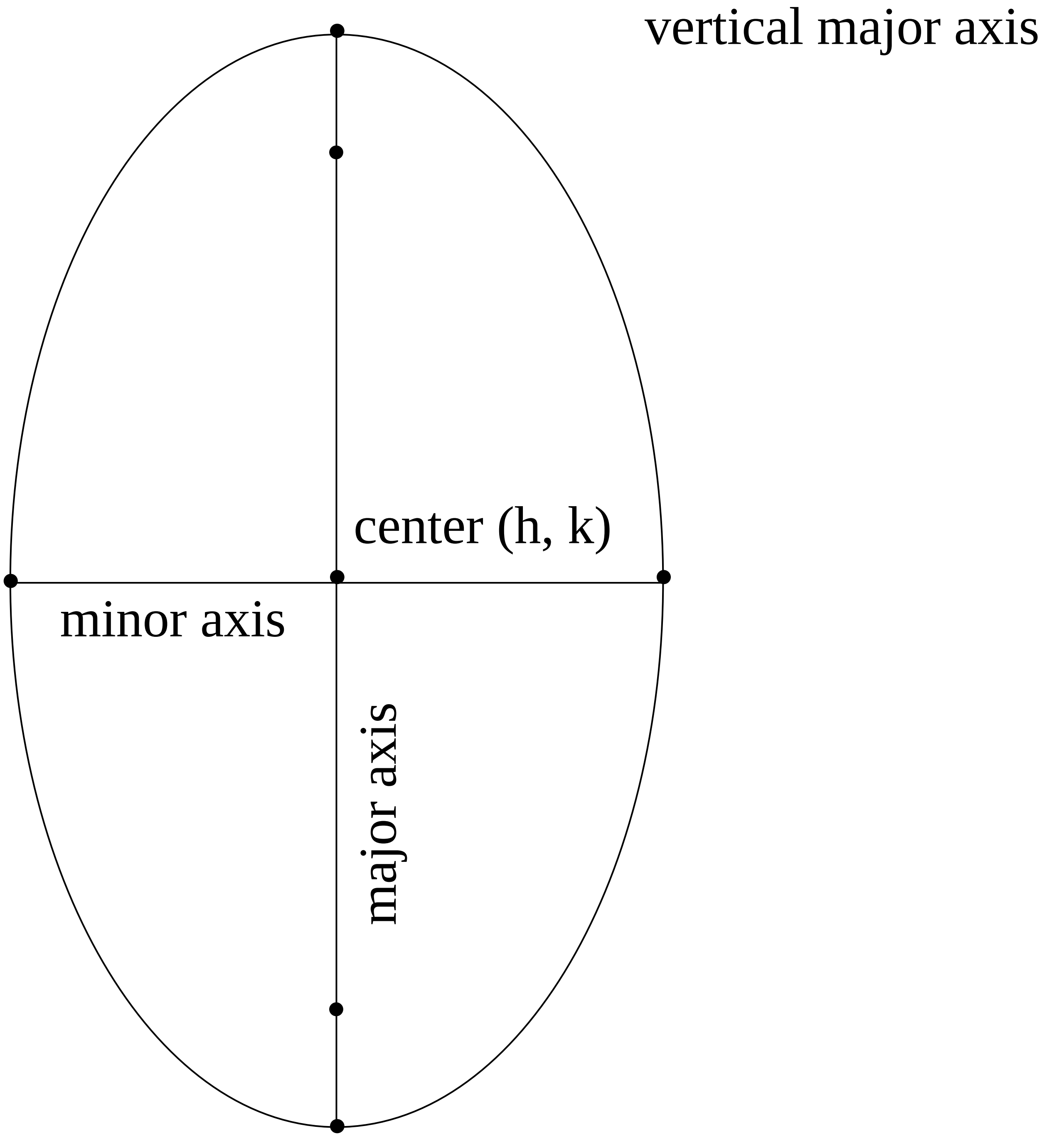
Domain= {x: h - a ≤ x ≤ h + a}

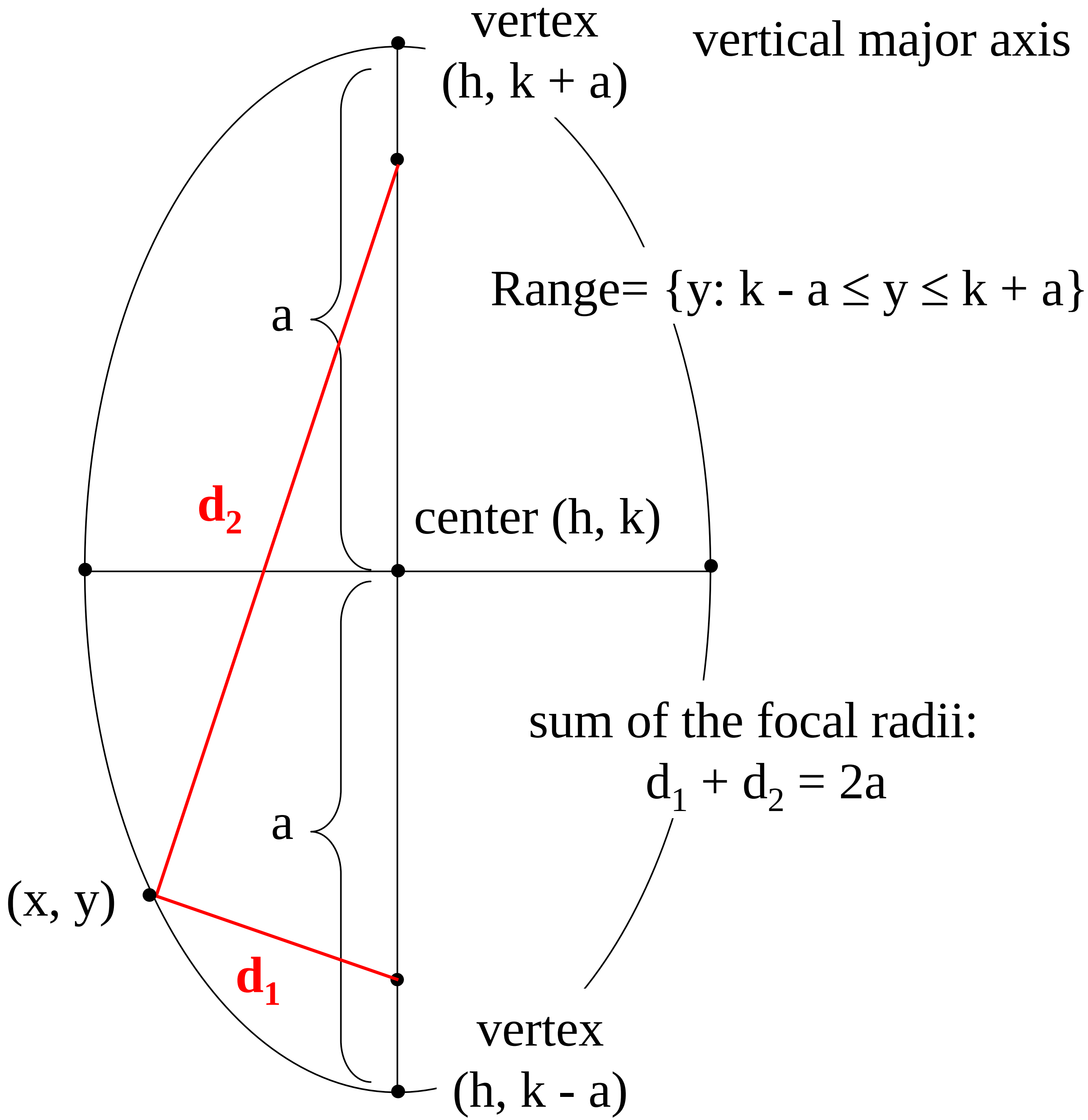
horizontal major axis

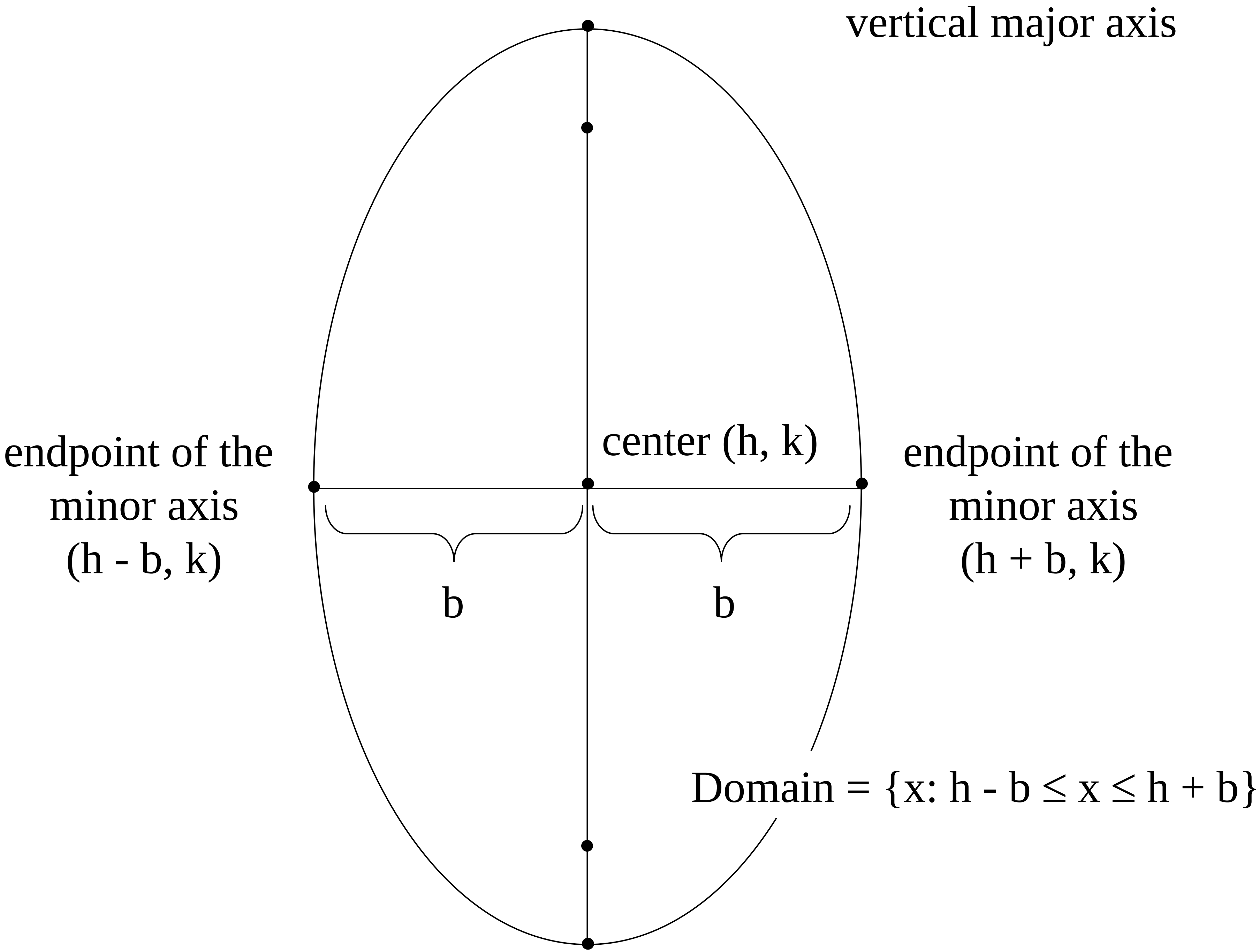


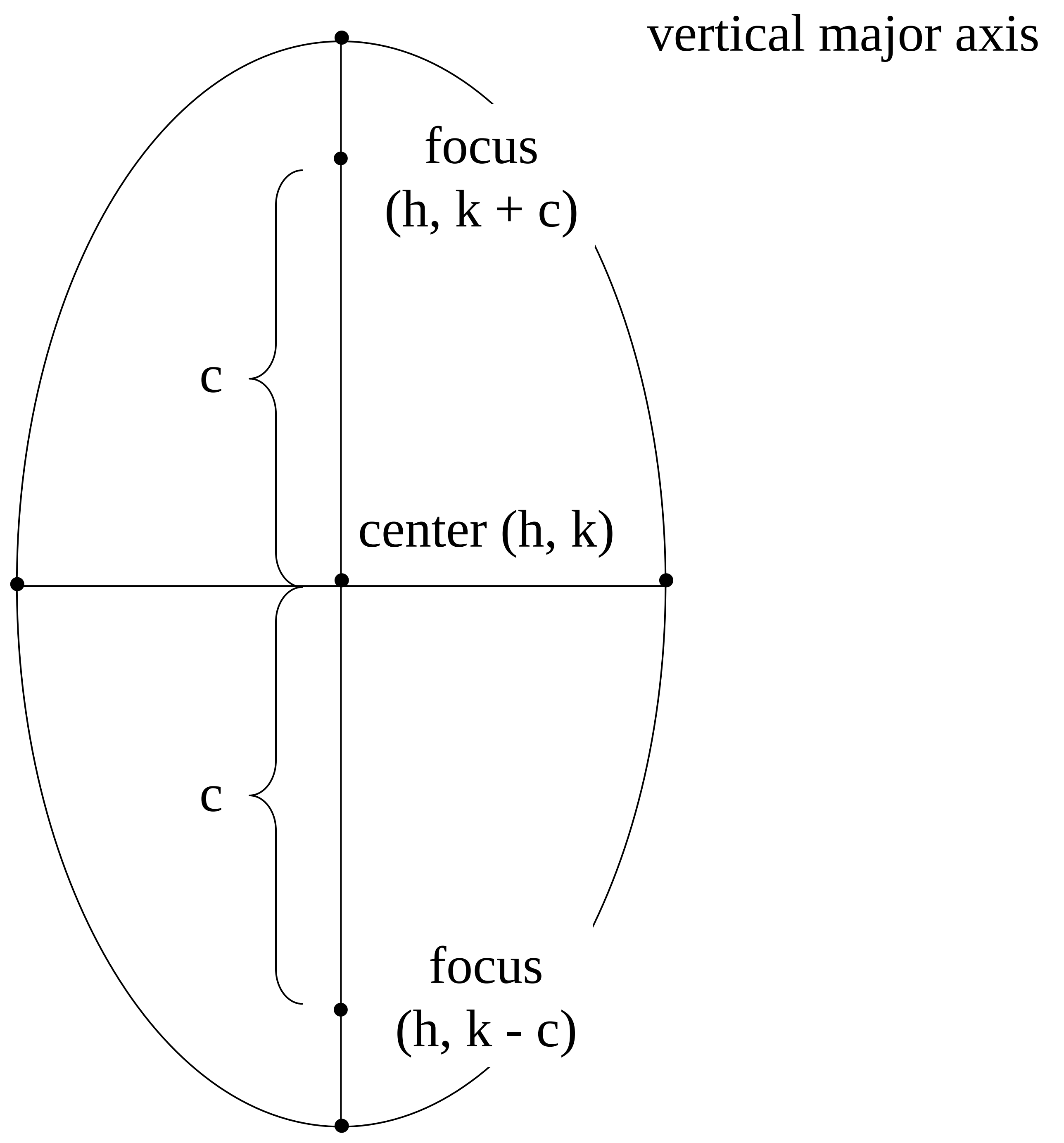
horizontal major axis



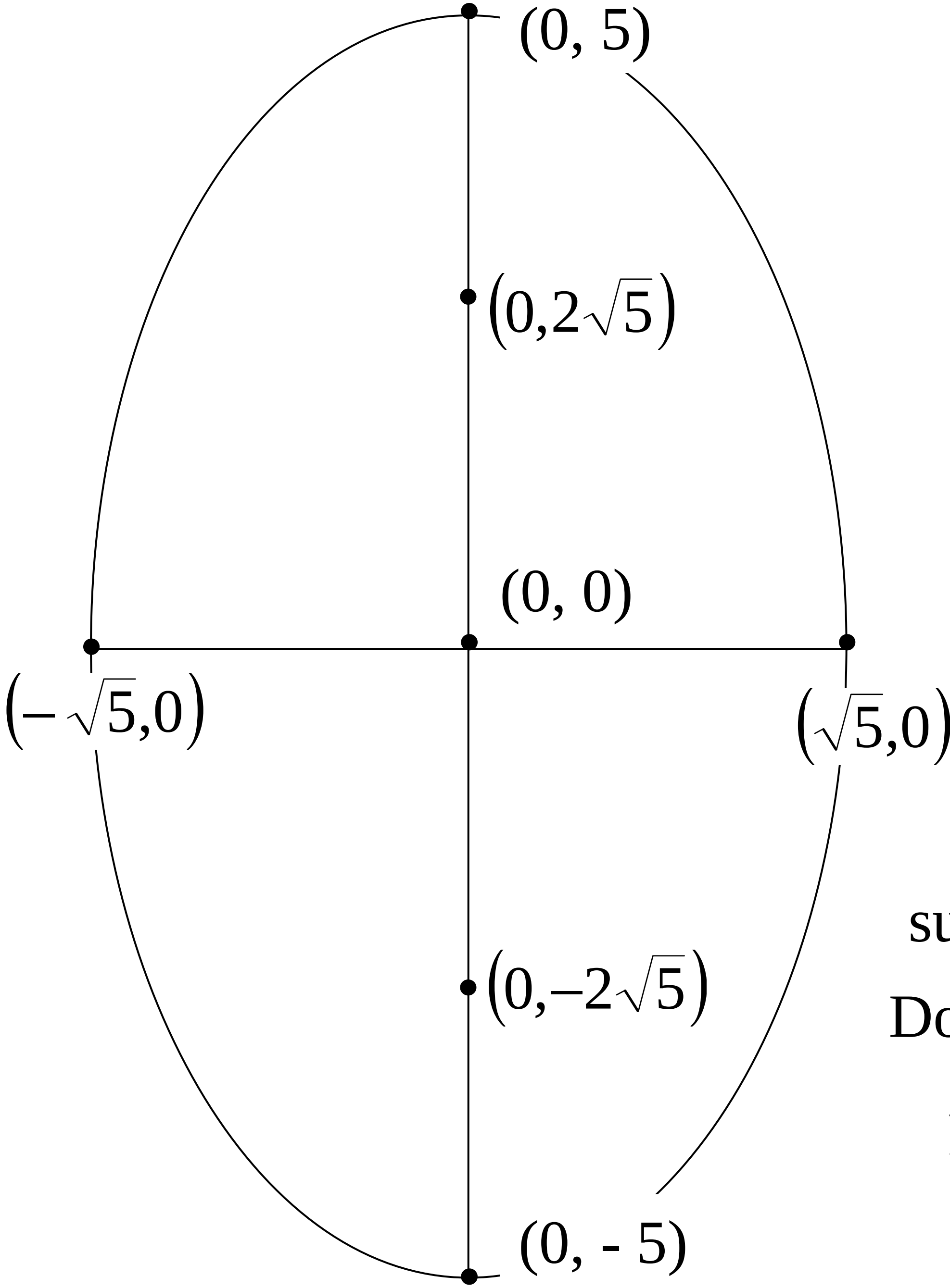

vertical major axis
center (h, k)
minor axis
major axis


vertex
(h, k + a)
vertical major axis
Range= {y: k - a ≤ y ≤ k + a}
a
d2
center (h, k)
sum of the focal radii:
d1 + d2 = 2a
a
(x, y)
d1
vertex
(h, k - a)


vertical major axis
center (h, k)
endpoint of the minor axis (h - b, k)
endpoint of the minor axis (h + b, k)
b
b
Domain = {x: h - b ≤ x ≤ h + b}


vertical major axis
focus
(h, k + c)
c
center (h, k)
c
focus
(h, k - c)

Example for 1-14



$5x^2 + y^2 = 25$

$$\frac{x^2}{5} + \frac{y^2}{25} = 1$$

$$\frac{(x-0)^2}{5} + \frac{(y-0)^2}{25} = 1$$

$a = 5$

$b = \sqrt{5}$

$c = 2\sqrt{5}$

sum of the focal radii = 10

Domain: $\{x : -\sqrt{5} \le x \le \sqrt{5}\}$

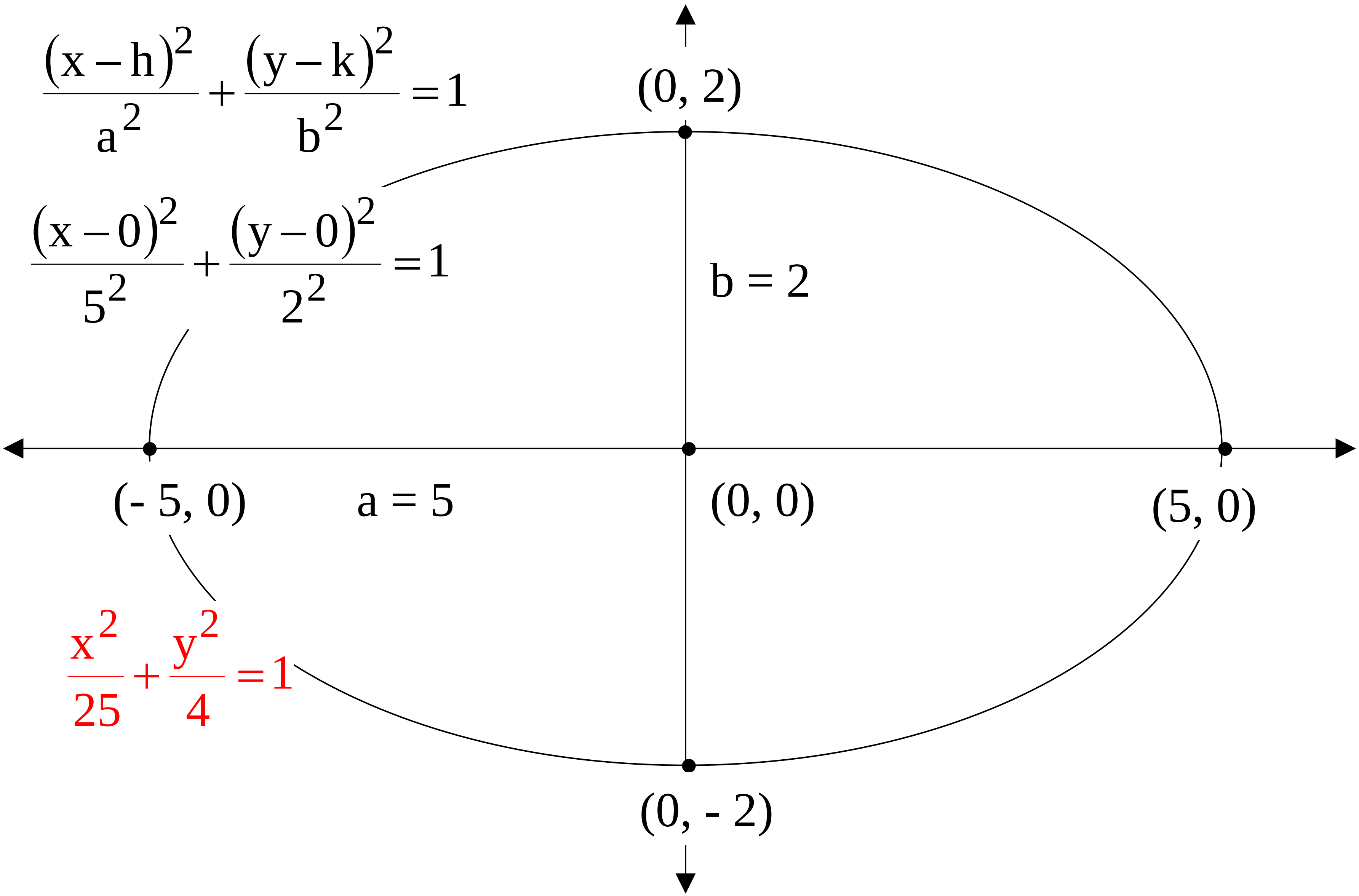
Range: $\{y : -5 \le y \le 5\}$

Example for 15-18

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$$\frac{(x-0)^2}{5^2}+\frac{(y-0)^2}{2^2}=1$$

$$\frac{x^2}{25}+\frac{y^2}{4}=1$$



Example for 19-22

$2a = 18$

$a = 9$

$36 = 81 - b^2$

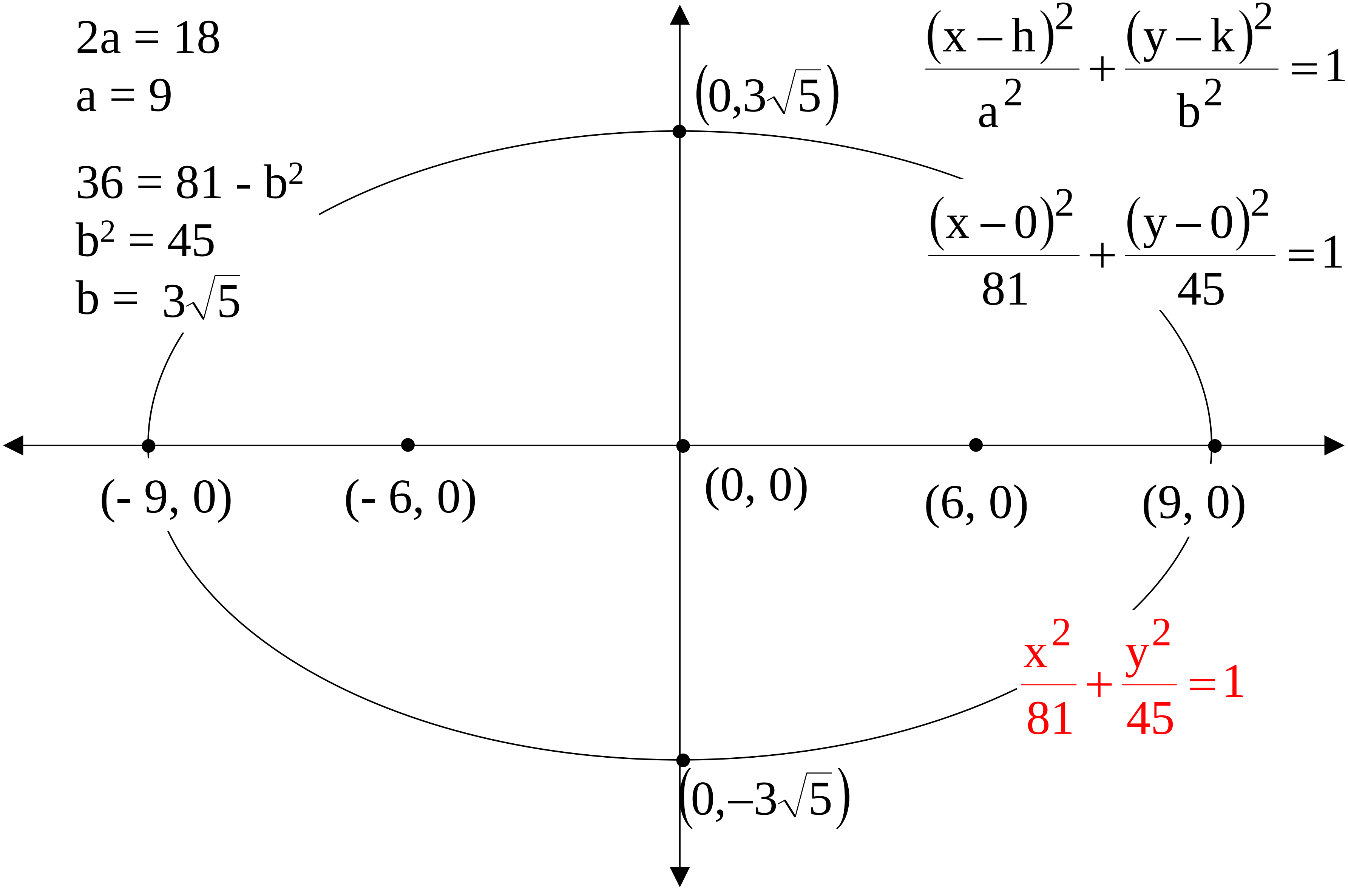
$b^2 = 45$

$b = 3\sqrt{5}$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-0)^2}{81} + \frac{(y-0)^2}{45} = 1$$

$$\frac{x^2}{81} + \frac{y^2}{45} = 1$$



Example for 28-31

- Graph as you did in 1-14 using the rules for inequalities when you draw your ellipse.
- Substitute in the center. If it makes a true statement, then shade the inside of the ellipse. If it makes a false statement, then shade the outside of the ellipse.

Section 9-5

Hyperbolas

Identify and graph all 12 characteristics

Objectives

- to identify and graph the characteristics of a hyperbola
- to write the standard form equation of a hyperbola
- to graph hyperbolic inequalities

Standard Form for a Hyperbola

- A hyperbola is the set of all points in the plane such that the difference between the distances from a point P to two fixed points (foci) is a given constant (difference of the focal radii).
- Standard Form Equation of a Hyperbola: center (h, k); c = $\sqrt{a^2 + b^2}$
  - horizontal axis of symmetry

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

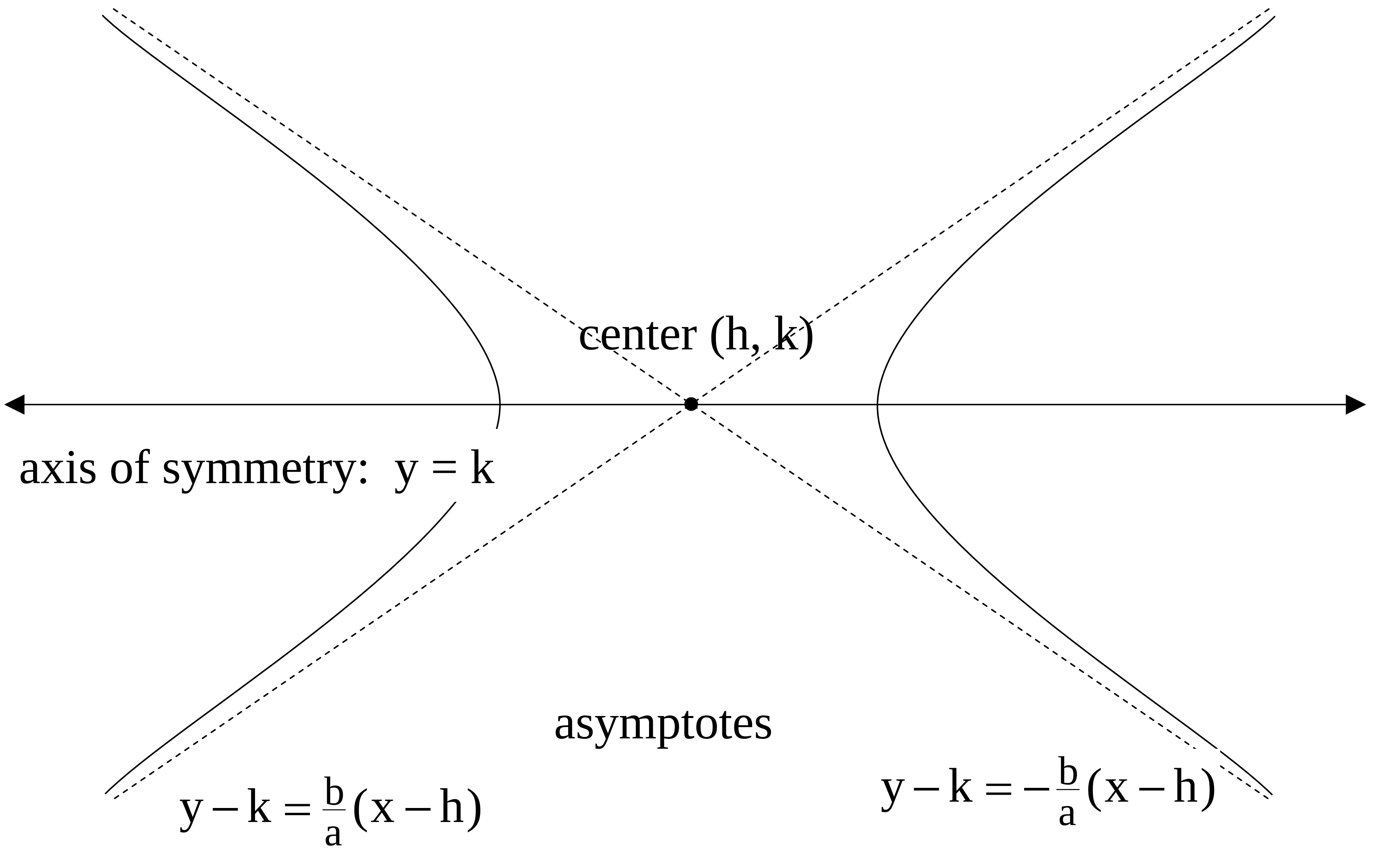
  - vertical axis of symmetry

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

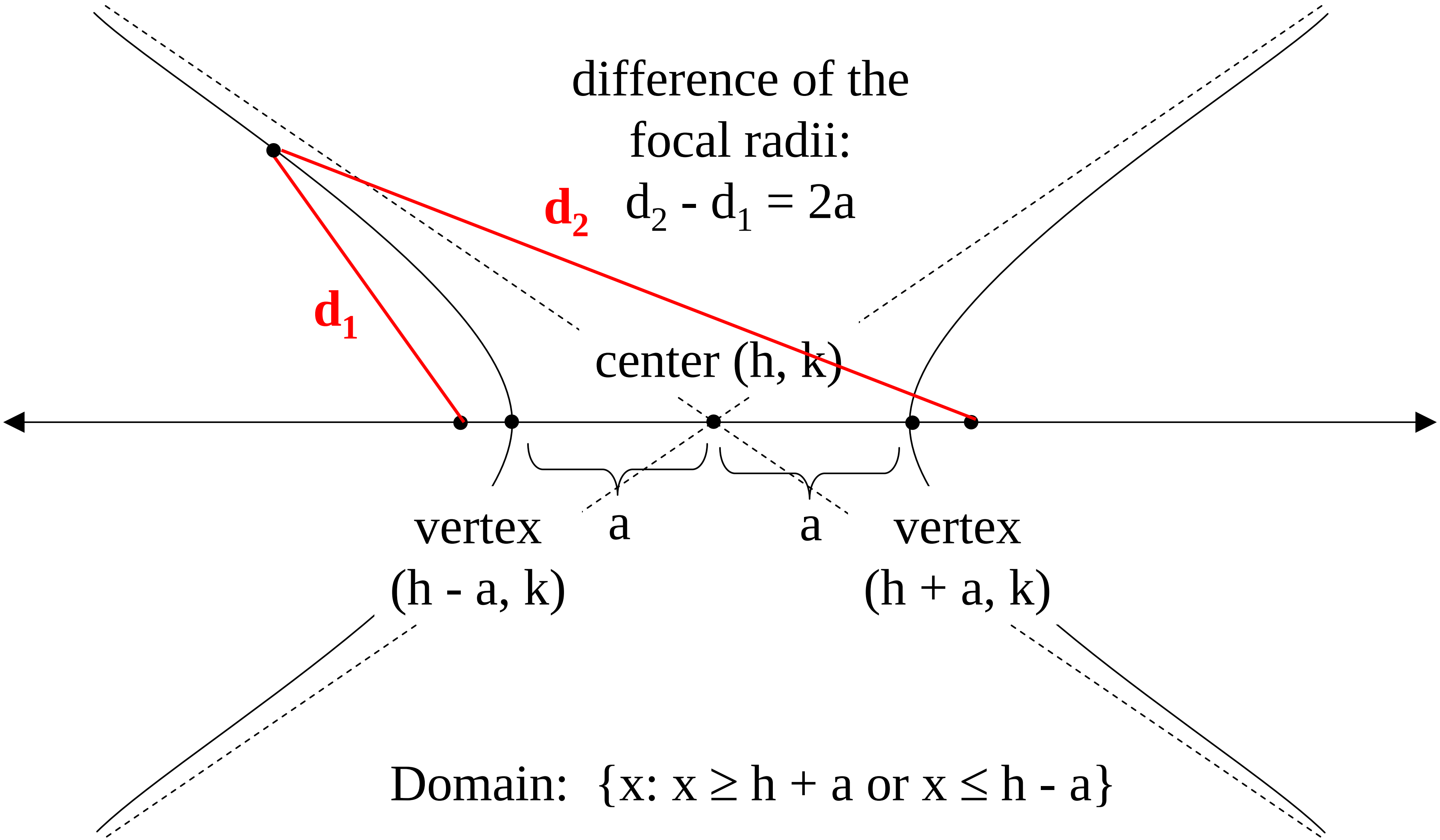
12 Characteristics of a Hyperbola

1. value of a: distance between the center and the vertices
2. value of b
3. value of c: distance between the center and the foci
4. center: (h, k)
5. direction: horizontal/vertical axis of symmetry
6. equation of the axis of symmetry
7. equations of the asymptotes
8. coordinates of vertices
9. coordinates of foci
10. difference of the focal radii
11. domain
12. range

horizontal axis of symmetry



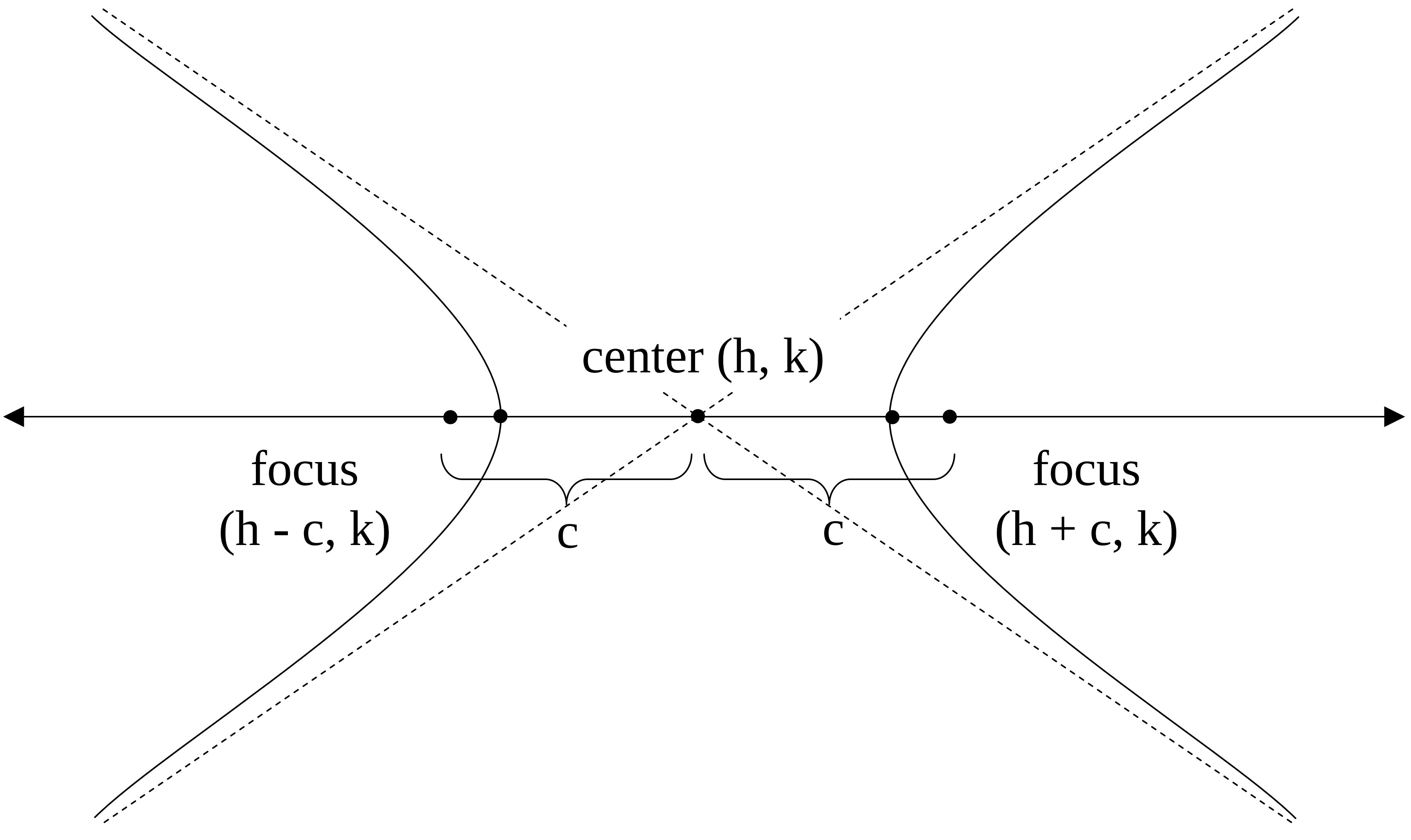
horizontal axis of symmetry

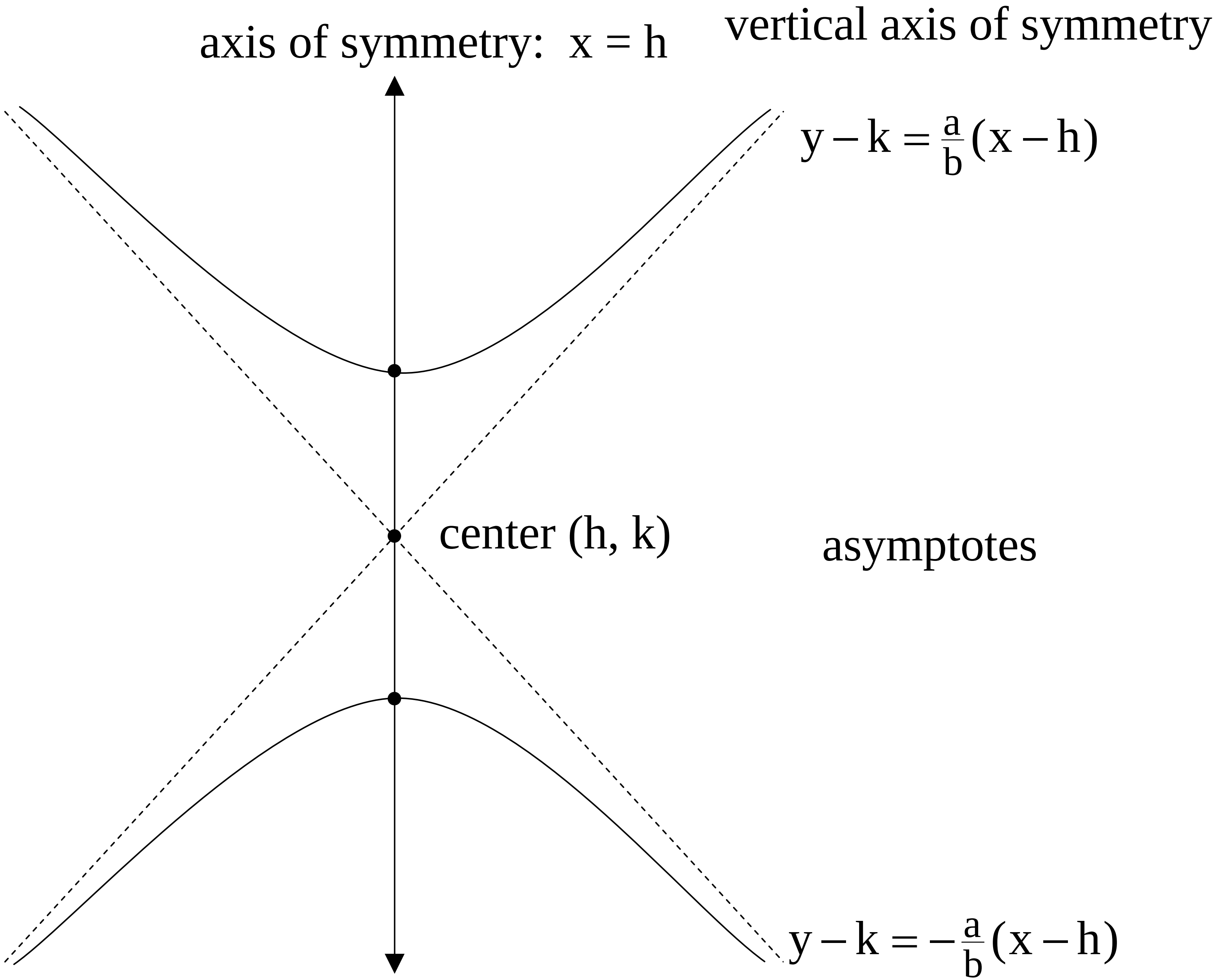


Domain: $\{x: x \geq h + a \text{ or } x \leq h - a\}$

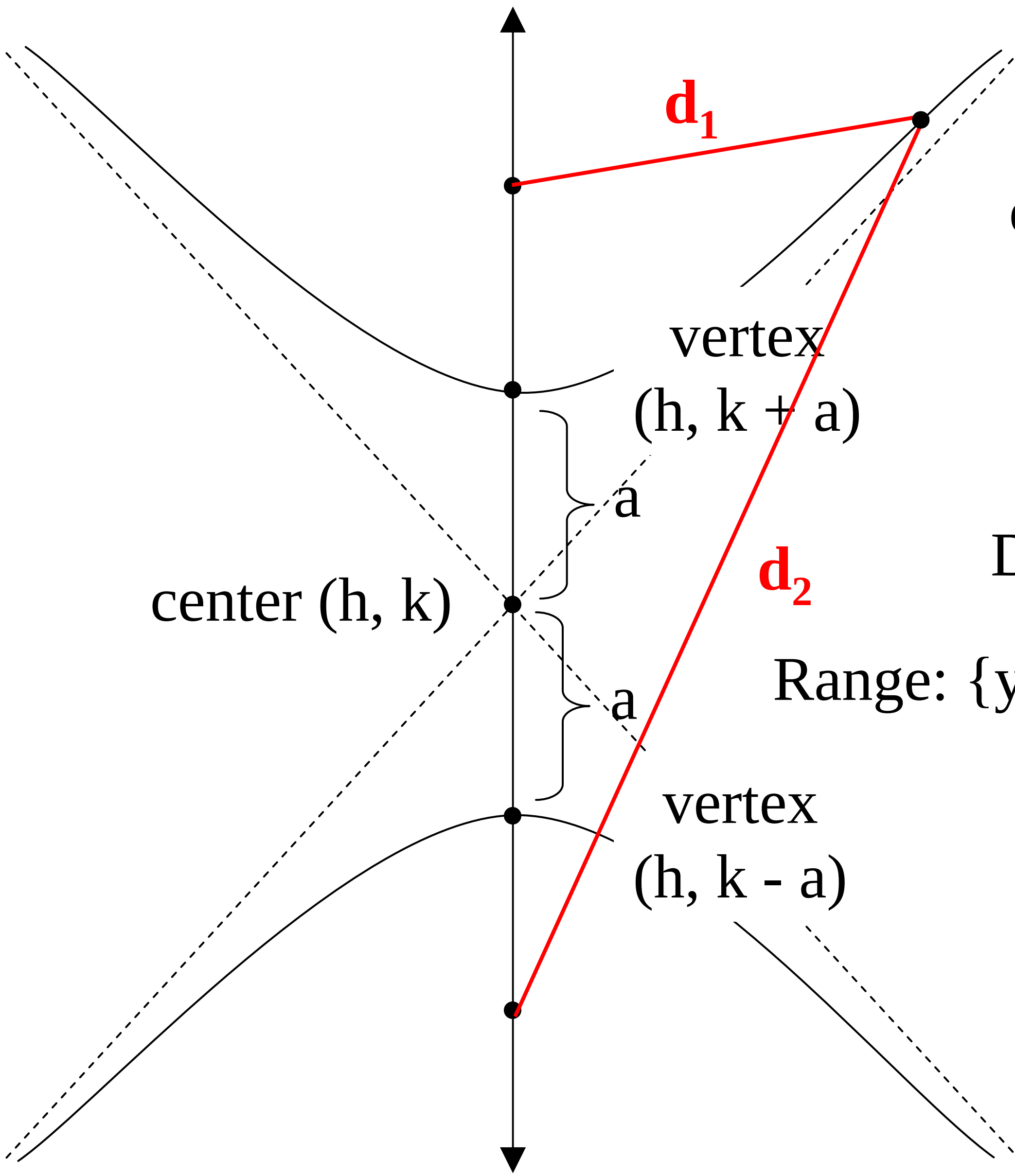
Range: $\{y: y \in \Re\}$

horizontal axis of symmetry




vertical axis of symmetry
axis of symmetry: x = h
$y - k = \frac{a}{b}(x - h)$
center (h, k)
asymptotes
$y - k = -\frac{a}{b}(x - h)$

vertical axis of symmetry



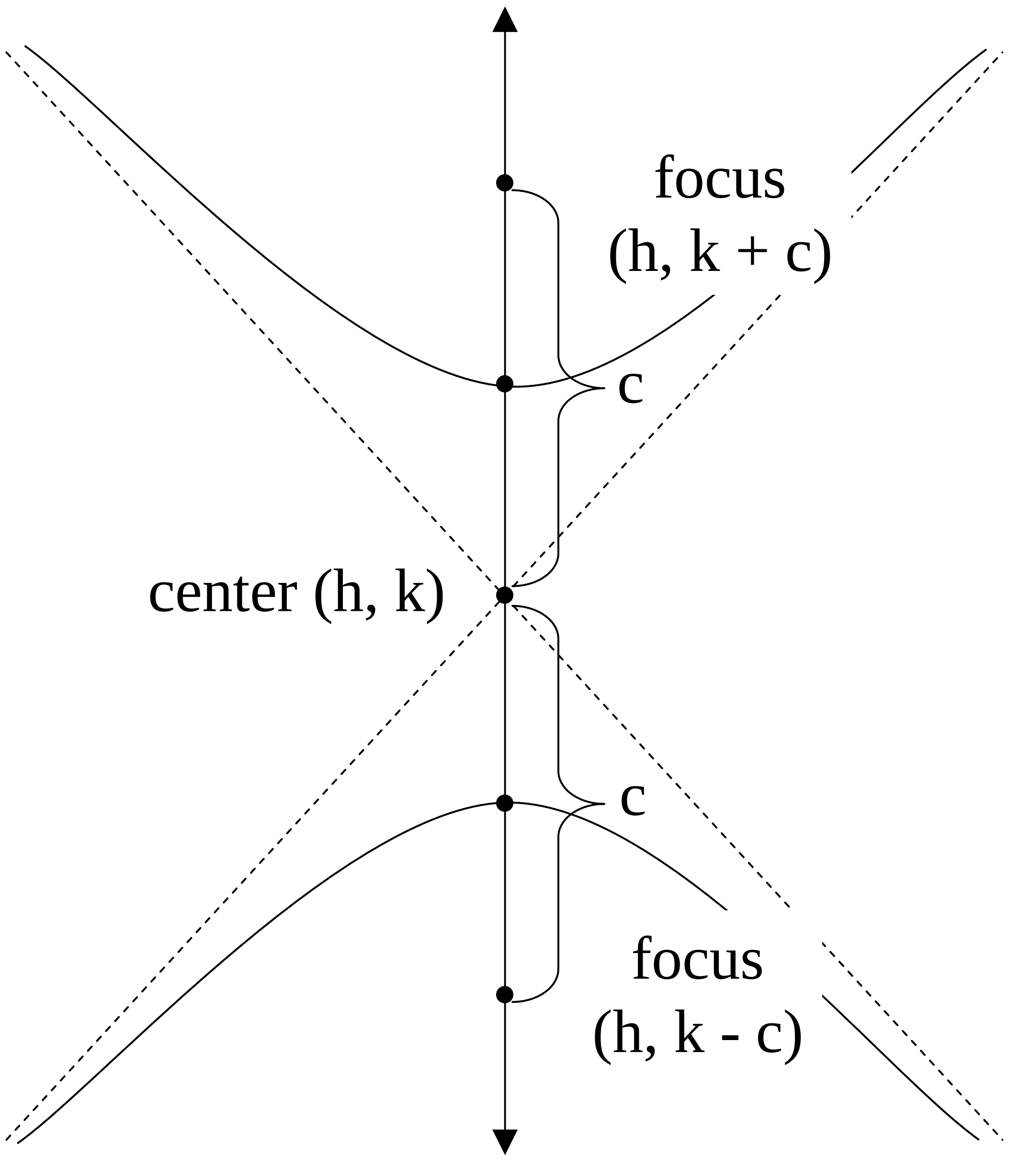
difference of the focal radii:

$d_2 - d_1 = 2a$

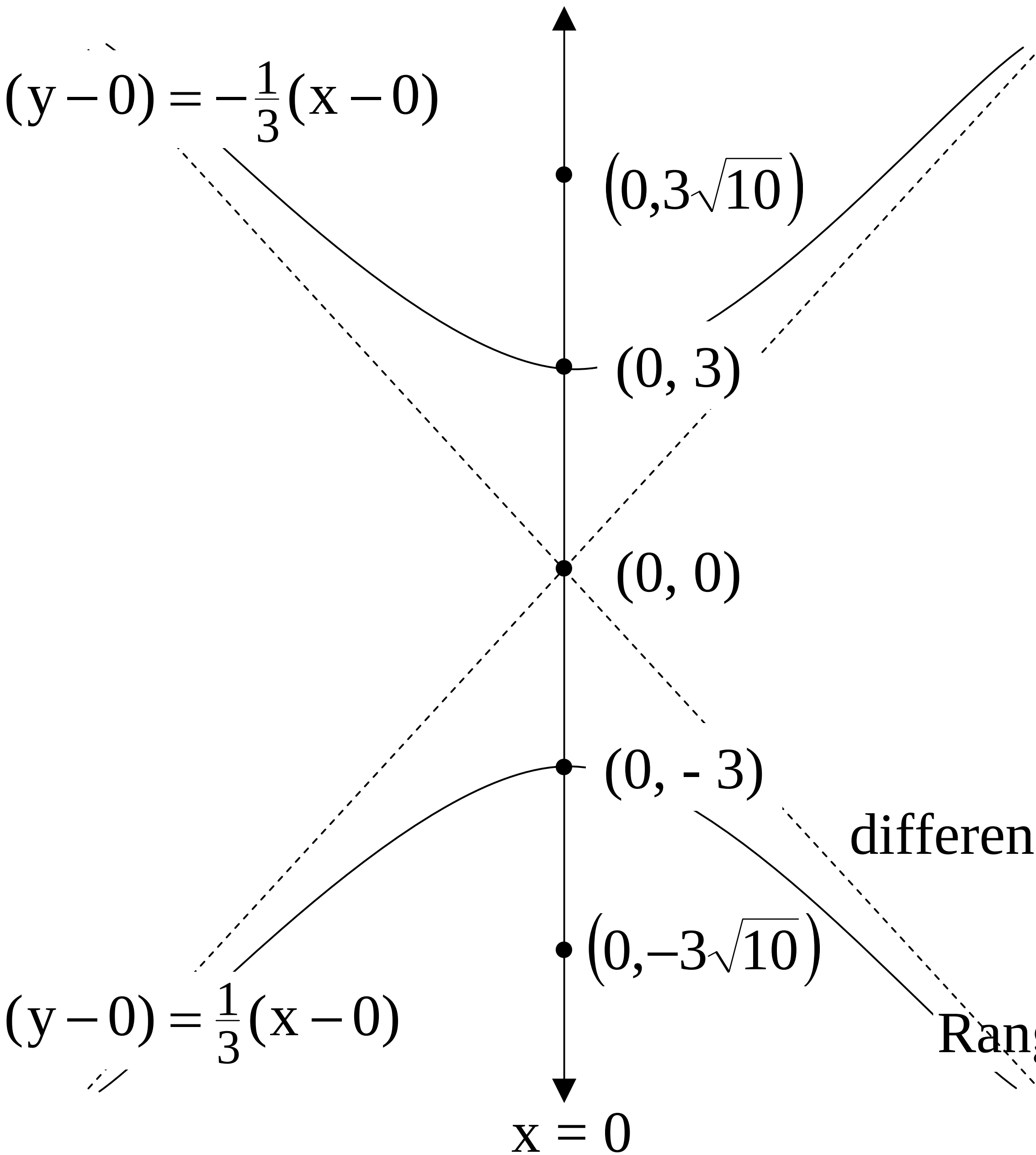
Domain: $\{x: x \in \Re\}$

Range: $\{y: y \geq k + a \text{ or } y \leq k - a\}$

vertical axis of symmetry



Example for 1-12



$$x^2 = 9y^2 - 81$$

$$9y^2 - x^2 = 81$$

$$\frac{y^2}{9} - \frac{x^2}{81} = 1$$

$$\frac{(y-0)^2}{9} - \frac{(x-0)^2}{81} = 1$$

$$a = 3$$

$$b = 9$$

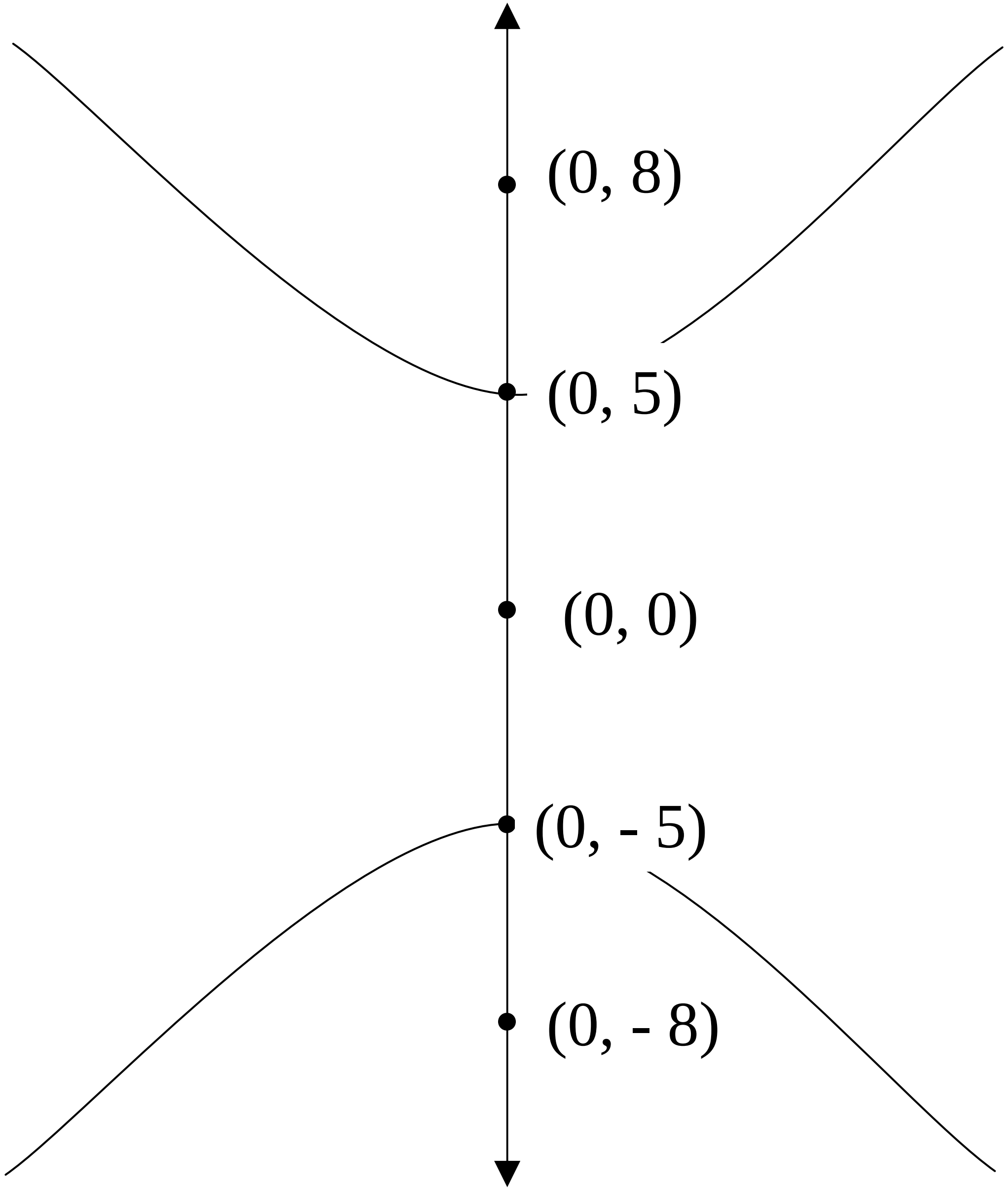
$$c = 3\sqrt{10}$$

difference of the focal radii = 6

Domain: $\{x : x \in \Re\}$

Range: $\{y : y \geq 3 \text{ or } y \leq -3\}$

Example for 13-18



$2a = 10$

$a = 5$

$64 = 25 + b^2$

$39 = b^2$

$\sqrt{39} = b$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{(y-0)^2}{25} - \frac{(x-0)^2}{39} = 1$$

$$\frac{y^2}{25} - \frac{x^2}{39} = 1$$

Example for 19-22

- Graph as you did in 1-12 using the rules for inequalities when you sketch the hyperbola.
- Substitute the center into the inequality. If it makes a true statement, then shade outside the hyperbola. If it makes a false statement, then shade inside the hyperbola.

Example for 30 & 31

- Solve the same as the example for 13-18.

Section 9-6

Conic Sections

Identify and graph all of the appropriate characteristics

Objectives

- to find the standard form equation of an ellipse
- to find the standard form equation of a hyperbola
- to identify a conic section from its equation, convert to standard form by completing the square, and then to identify and graph all of its characteristics

Identifying conics from their Non-Standard Form

- If only one variable is squared, then the equation is for a parabola.
  - opening up/down if only the x is squared
  - opening left/right if only the y is squared
- If both x and y are squared, they are both positive and they have the same coefficient, then the equation is for a circle.
- If both the x and the y are squared, they are both positive and they have different coefficients, then the equation is for an ellipse.
- If both the x and the y are squared and one is positive and the other is negative, then the equation is for a hyperbola.
- **To convert to a Standard Form Conic Equation you must complete the square for any quadratic variables.**

Example for 1-6

- Solve the same as 19-22 in section 9-4 except that the center is not (0, 0) it is the midpoint of the foci and c = half the distance between the foci.

Example for 7-12

- Solve the same as 30-31 in section 9-5 except that the center is not (0, 0) it is the midpoint of the foci and c = the half the distance between the foci.

Example for 13-18

$x^2 - 4y^2 - 2x - 24y - 39 = 0$

$(x^2 - 2x + 1) - 4(y^2 + 6y + 9) = 39 + 1 - 36$

$(x - 1)^2 - 4(y + 3)^2 = 4$

$$\frac{(x-1)^2}{4} - \frac{(y+3)^2}{1} = 1$$

$a = 2$

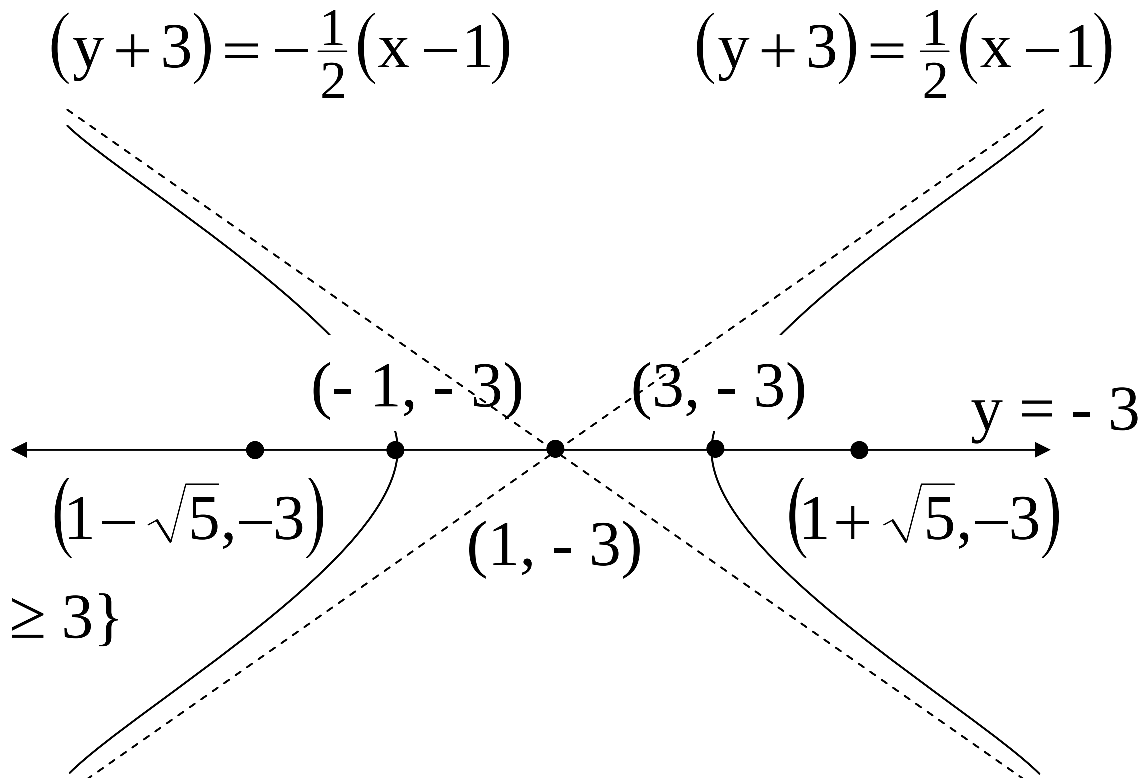
$b = 1$

$c = \sqrt{5}$

difference of the focal radii = 4

Domain $\{x: x \leq -1 \text{ or } x \geq 3\}$

Range $\{y: y \in \Re\}$



Example for 19 & 20

- Solve the same as 19-22 in section 9-4 except that the center is not (0, 0) it is the midpoint of the foci and c = half the distance between the foci.

Section 9-7

Quadratic Systems

Objectives

- to identify the number of solutions to a quadratic system
- to graph the solution set to a system of quadratic inequalities

Systems involving conic equations/inequalities

- Systems involving conic equations can have either four, three, two, one or zero solutions (intersections).
- It is not necessary to graph all of the characteristics of a conic when solving a system:
  - parabola: graph its vertex, intercepts and direction
  - circle: graph its center and four points on the circle at the ends of the horizontal and vertical diameters
  - ellipse: graph its center, vertices and the endpoints of the minor axis
  - hyperbola: graph its center, vertices and asymptotes
- Systems of conic inequalities have overlapping regions as their solutions. Follow the same rules for graphing conic inequalities and systems as you did for linear inequalities and systems.

Example for 1-9

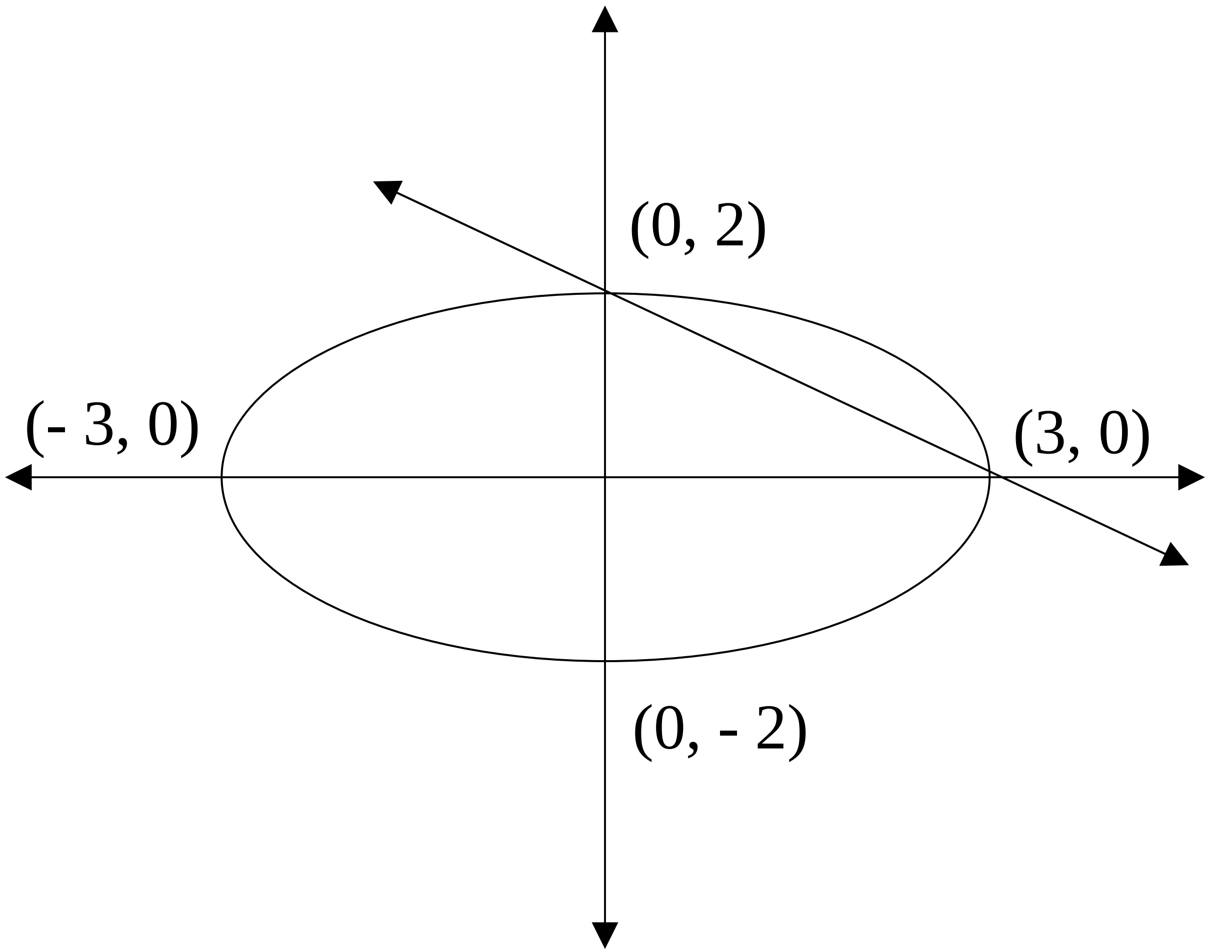
$4x^2 + 9y^2 = 36$

$2x + 3y = 6$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

$$2x + 3y = 6$$

two

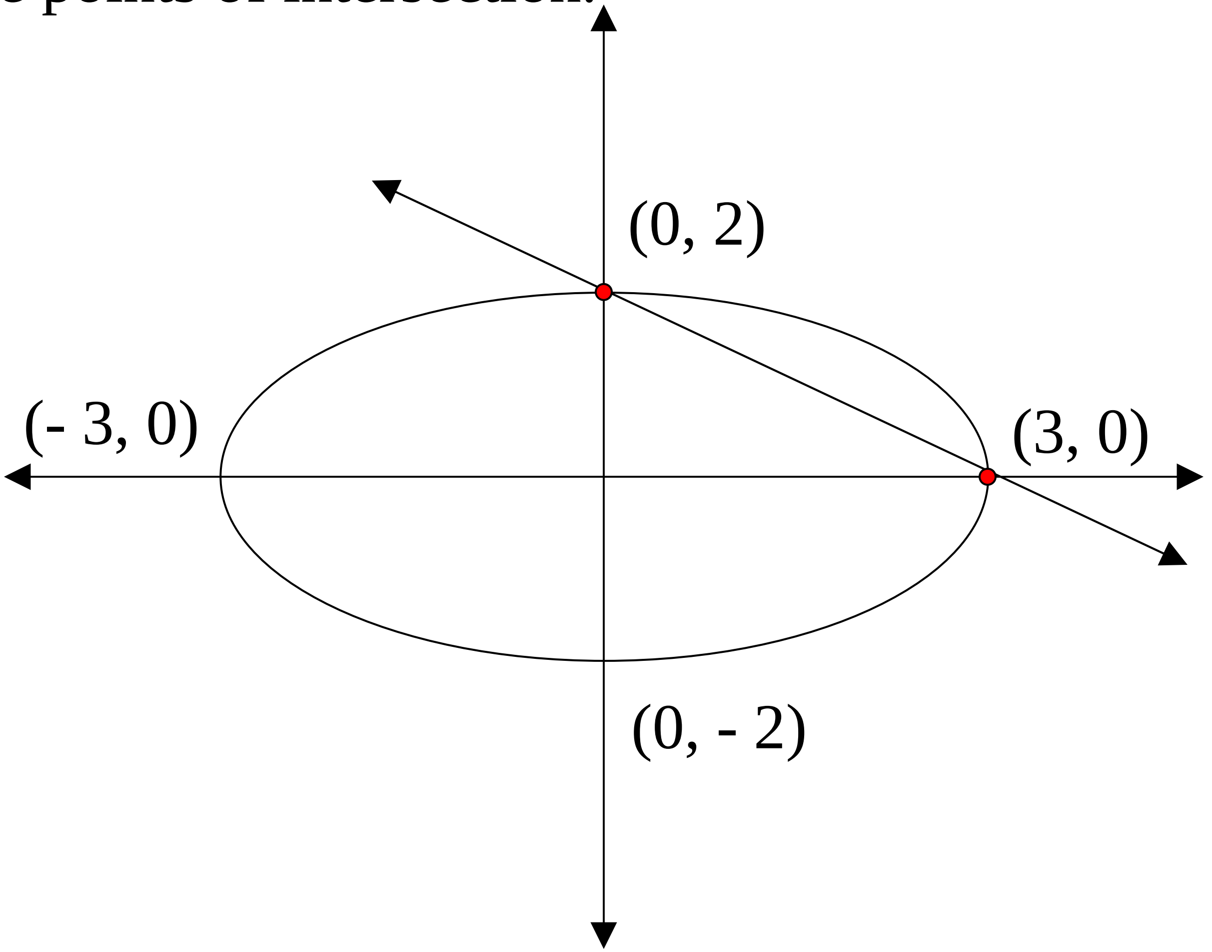


Example for 10-15

- Use your graphs from 4-9 to estimate to the nearest half unit the coordinates of the intersections. If your graphs are accurate enough you may be able to identify the exact coordinates of the points of intersection.

$4x^2 + 9y^2 = 36$

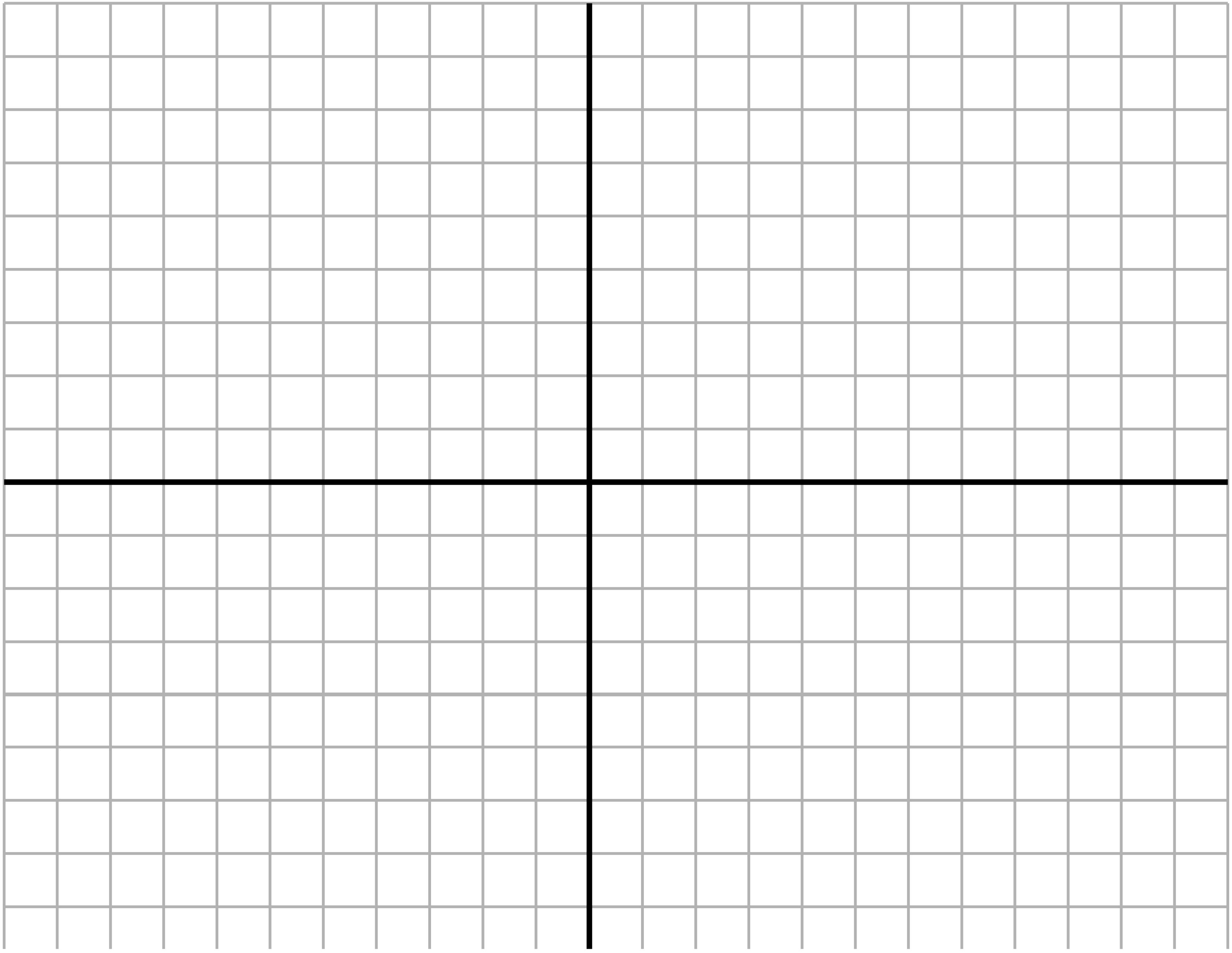
$2x + 3y = 6$



(0, 2) & (3, 0)

Example for 16-19

- Graph each inequality as you have in the last several sections (9-2: 25-30; 9-3: 27-30; 9-4: 28-31; 9-5: 19-22).
- Your answer is where the shaded regions overlap.



Section 9-8

Quadratic Systems

Objectives

- to solve quadratic systems

Solving systems of conic equations

- You may use the same methods to solve a system involving conic equations as you did to solve systems of linear equations:
  - add
  - subtract
  - multiply then add/subtract
  - substitution
  - graphing
- Answers must be in exact form.
- Imaginary solutions will not show up from a graph.

Example for 1-18

$x^2 + y^2 = 25$
$x^2 - y^2 = 7$

$x^2 = y^2 + 7$

$y^2 + 7 + y^2 = 25$

$2y^2 = 18$

$y^2 = 9$

$y = \pm 3$

$x^2 = (3)^2 + 7$

$x^2 = 9 + 7$

$x^2 = 16$

$x = \pm 4$

(4, 3) & (- 4, 3)

$x^2 = (- 3)^2 + 7$

$x^2 = 9 + 7$

$x^2 = 16$

$x = \pm 4$

(4, - 3) & (- 4, - 3)

(4, 3) (- 4, 3) (4, - 3) (- 4, - 3)

Example for 19 & 20

$y^2 = x + 7$
$xy = 6$

$x = \frac{6}{y}$

$y^2 = \frac{6}{y} + 7$

$y^3 = 6 + 7y$

$y^3 - 7y - 6 = 0$

one positive root
2 negative roots
or
one positive root
2 imaginary roots

$\{\pm 1, \pm 2, \pm 3, \pm 6\}$

- 2	1	0	- 7	- 6
		- 2	4	6
	1	- 2	- 3	0

$y^2 - 2y - 3 = 0$
$(y - 3)(y + 1) = 0$

$y = -2, -1, 3$

(- 3, - 2) (- 6, - 1)
(2, 3)

Section 9-9

Linear Systems in Three Variables

Objectives

- to solve linear systems in three variables i.e. finding intersections of planes in 3 dimensional space

Linear Equations in three variables

- Linear equations in three variables represent planes in three dimensional space.
- Systems of linear equations in three variables have three possible solutions;
  - one point (x, y, z) in three dimensional space
  - one line in the x-y plane, y-z plane or the x-z plane
  - no intersection because two or all of the planes are parallel
- Systems in three variables are solved the same way two variable systems are solved except that the process must be repeated twice

Example for 1-18

(I) $2x + 3y = 6 + z$
(II) $x - 2y = -1 - z$
(III) $3x + y = -1 + 3z$

(I) $2x + 3y - z = 6$
(II) $x - 2y + z = -1$
(III) $3x + y - 3z = -1$

(I) $2x + 3y - z = 6$
(II) $\underline{x - 2y + z = -1}$
(IV) $3x + y = 5$

(II) $3[x - 2y + z = -1]$
(III) $\underline{3x + y - 3z = -1}$

(II) $3x - 6y + 3z = -3$
(III) $\underline{3x + y - 3z = -1}$
(V) $6x - 5y = -4$

(IV) $5[3x + y = 5]$
(V) $\underline{6x - 5y = -4}$

(IV) $15x + 5y = 25$
(V) $\underline{6x - 5y = -4}$
$21x = 21$
$x = 1$

(V) $6(1) - 5y = -4$
$-5y = -10$
$y = 2$

(I) $2(1) + 3(2) - z = 6$
$2 + 6 - z = 6$
$8 - z = 6$
$z = 2$

(1, 2, 2)

Example for 19 & 20

(I) $x - 2y + 3z = 1$
(II) $x + y - 3z = 7$
(III) $3x - 4y + 5z = 7$

(I) $x - 2y + 3z = 1$
(II) $\underline{x + y - 3z = 7}$
(IV) $-3y + 6z = -6$

(I) $3[x - 2y + 3z = 1]$
(III) $\underline{3x - 4y + 5z = 7}$

(I) $3x - 6y + 9z = 3$
(III) $\underline{3x - 4y + 5z = 7}$
(V) $-2y + 4z = -4$

(IV) $-3y + 6z = -6$
(V) $\underline{-2y + 4z = -4}$

(IV) $-y + 2z = -2$
(V) $\underline{-y + 2z = -2}$

$2z + 2 = y$

(II) $x + (2z + 2) - 3z = 7$

$x = z + 5$

$(z + 5, 2z + 2, z)$