Section 11-1

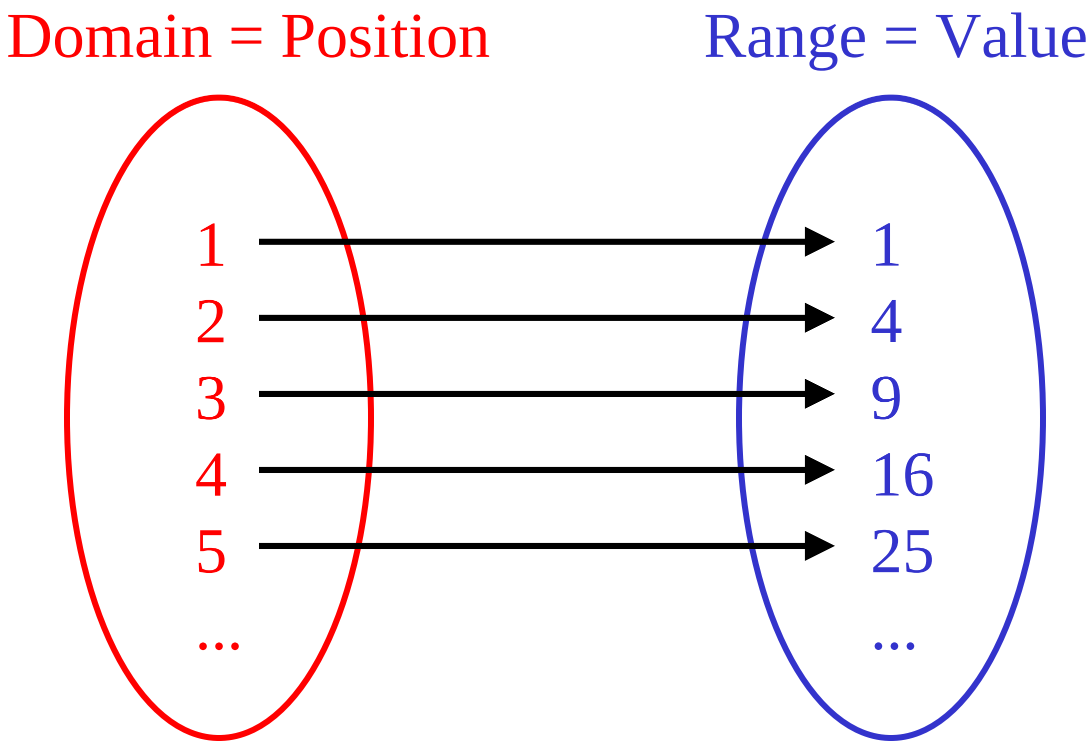
Types of Sequences

Objectives

- to determine whether a sequence is arithmetic, geometric or neither
- to find the first four terms in a sequence
- to find the next two terms in a sequence
- to identify a sequence from a diagram

Sequences

- sequence: a function whose domain is the set of whole numbers, representing position in a list, and whose range is a set of terms, representing values in the list.



Arithmetic Sequence/Arithmetic Progression

- a list of terms with a common difference (d)
- If $t_n - t_{n-1} = t_{n+1} - t_n$ then the sequence has a common difference d and is arithmetic.
- $d = t_n - t_{n-1}$

Geometric Sequence/Geometric Progression

- a list of terms with a common ratio (r)
- If $\frac{t_n}{t_{n-1}} = \frac{t_{n+1}}{t_n}$ then the sequence has a common ratio and is geometric.
- $r = \frac{t_n}{t_{n-1}}$

Examples for 1-10

20, 17, 14, 11, ___, ___

17 - 20 = - 3

14 - 17 = - 3

11 - 14 = - 3

The terms have a common difference d = - 3; therefore the sequence is **arithmetic**.

The missing terms are **8, 5**

Examples for 11-18

$$t_n = 3^{n-1}$$

$$t_1 = 3^{1-1} = 3^0 = 1$$

$$t_2 = 3^{2-1} = 3^1 = 3$$

$$t_3 = 3^{3-1} = 3^2 = 9$$

$$t_4 = 3^{4-1} = 3^3 = 27$$

1, 3, 9, 27; Geometric

Examples for 19 & 20

- 3, - 1, 1, 3, ...

- 1 - (- 3) = 2

1 - (- 1) = 2

3 - 1 = 2

The sequence has a common difference of 2; therefore, it is **arithmetic**.

$$2^{-3}, 2^{-1}, 2^{1}, 2^{3}, \ldots$$

$$\frac{1}{2} - \frac{1}{8} = \frac{3}{8}$$

$$2 - \frac{1}{2} = \frac{3}{2}$$

It does not have a common difference; therefore, it is not arithmetic.

$$\frac{2^{-1}}{2^{-3}} = 2^{2} = 4$$

$$\frac{2}{2^{-1}} = 2^{2} = 4$$

$$\frac{2^{3}}{2} = 2^{2} = 4$$

The sequence has a common ratio of 4; therefore, it is **geometric**.

Examples for 21-30

sequence:	2	4	8	14	22	**32**	**44**
difference of terms:	2	4	6	8	10	12	

The sequence is neither arithmetic nor geometric; however, the sequence created by the difference in the terms is arithmetic with a common difference of 2.

Section 11-2

Arithmetic Sequences

Objectives

- to find the formula for the nth term in an arithmetic sequence
- to find the specified term of an arithmetic sequence
- to find an arithmetic mean between values
- to find several arithmetic means between values
- to find the position of a term in an arithmetic sequence

Arithmetic Sequences

- In an arithmetic sequence with the first term t_1 and common difference d, the nth term (general) term is given by $t_n = t_1 + (n - 1)d$.
- Any terms between two values in an arithmetic sequence are called arithmetic means.

Examples for 1-6

24, 32, 40, 48, ...

$t_n = t_1 + (n - 1)d$

$d = 32 - 24 = 8$

$t_1 = 24$

$t_n = 24 + (n - 1)8$

$t_n = 24 + 8n - 8$

$\mathbf{t_n = 8n + 16}$

Examples for 7-18

$t_7 = -19,\ \ t_{10} = -28,\ t_{21}$

$t_n = t_1 + (n - 1)d$

$t_{21} = t_1 + 20d$

$t_{10} = -28 = t_1 + 9d$

$t_7 = -19 = t_1 + 6d$

$-9 = 3d$

$d = -3$

$t_1 = -1$

$t_{21} = -1 + -60 =$ **- 61**

Examples for 19-22

$$\frac{-3+7}{2}=2$$

Examples for 23-26

$-27, 33$

2 arithmetic means	3 arithmetic means	4 arithmetic means
- 27, ___, ___, 33	- 27, ___, ___, ___, 33	- 27,___, ___, ___, ___, 33
$t_n = t_1 + (n - 1)d$	$t_n = t_1 + (n - 1)d$	$t_n = t_1 + (n - 1)d$
$33 = -27 + 3d$	$33 = -27 + 4d$	$33 = -27 + 5d$
$60 = 3d$	$60 = 4d$	$60 = 5d$
$d = 20$	$d = 15$	$d = 12$
- 7, 13	**- 12, 3, 18**	**- 15, - 3, 9, 21**

Examples for 27-30

18, 24, 30, ... , 618

$24 - 18 = 6 = d$

$t_n = t_1 + (n - 1)d$

$618 = 18 + (n - 1)6$

$618 = 18 + 6n - 6$

$618 = 12 + 6n$

$606 = 6n$

$n = 101$

Example for 31 & 32

$$25, 33, 41, \ldots, 145, \ldots$$

$$33 - 25 = 8 = d$$

$$t_n = t_1 + (n - 1)d$$

$$145 = 25 + (n - 1)8$$

$$145 = 25 + 8n - 8$$

$$145 = 17 + 8n$$

$$128 = 8n$$

$$\mathbf{n = 16}$$

Section 11-3

Geometric Sequences

Objectives

- to find the formula for the nth term in a geometric sequence
- to find the specified term in a geometric sequence
- to find the geometric mean between values
- to find several geometric means between values
- to determine which type of sequence a list of numbers represents and then to find the formula for the nth term
- to find formulas for sequences that are neither arithmetic nor geometric
- to solve applications of arithmetic and geometric sequences

Geometric Sequences

- In a geometric sequence with first term t_1 and common ratio r, the nth term (general) term is given by $t_n = (t_1)(r^{n - 1})$
- Any terms between two values in an geometric sequence are called geometric means.

Examples for 1-6

2, 6, 18, 54, ...

$$t_n = t_1 \bullet r^{n-1}$$

$$t_1 = 2$$

$$r = \frac{6}{2} = 3$$

$$t_n = 2 \bullet 3^{n-1}$$

Examples for 7-18

$$t_1 = 5, t_3 = 80; t_6$$

$$t_n = t_1 \bullet r^{n-1}$$

$$80 = 5 \bullet r^{3-1}$$

$$80 = 5 \bullet r^2$$

$$16 = r^2$$

$$r = 4$$

$$t_6 = 5 \bullet 4^{6-1} = 5 \bullet 4^5 = 5 \bullet 1024 = 5120$$

Examples for 19-22

2, 8

$$\text{mean} = \sqrt{(2)(8)}$$

mean = 4

Examples for 23-26

Three; 5, 80

5, ___, ___, ___, 80

$$t_n = t_1 \bullet r^{n-1}$$

$$80 = 5 \bullet r^{5-1}$$

$$80 = 5 \bullet r^{4}$$

$$16 = r^{4}$$

$$r = 2$$

5, 10, 20, 40, 80

Examples for 27-36

200, - 100, 50, - 25

- 100 - 200 = - 300
50- (- 100) = 150
not arithmetic

geometric

$$t_n = t_1 \bullet r^{n-1}$$

$$\frac{-100}{200} = -\frac{1}{2}$$

$$\frac{50}{-100} = -\frac{1}{2}$$

$$\frac{-25}{50} = -\frac{1}{2}$$

$$t_n = 200 \bullet \left(-\frac{1}{2}\right)^{n-1}$$

Examples for 37 & 38

$$\frac{2}{1}, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \ldots$$

The numerators form an arithmetic pattern.

$$t_n = t_1 + (n - 1)d$$

$$t_n = 2 + (n - 1)1$$

$$t_n = 2 + n - 1$$

$$t_n = n + 1$$

The denominators are neither arithmetic nor geometric, but they do have a pattern, position squared (n^2).

$$t_n = \frac{n+1}{n^2}$$

Section 11-4

Series and Sigma Notation

Objectives

- to write sigma notation in expanded form
- to write expanded series in sigma notation
- to find the missing summand of a series

Series and Sigma Notation

- When the terms of a sequence are added together the resulting expression is called a series.
- Arithmetic series: the related sequence is arithmetic.
- Geometric series: the related sequence is geometric.
- A series can be written in an abbreviated form known as sigma notation or summation notation.

$$\sum_{n=1}^{u} t_n$$

- t_n is called the summand. It is the formula for the general term in the series.
- n is called the index. The bottom number (lower limit) gives the position of the first term in the sum . The top number (upper limit) gives the position of the last term in the sum

Examples for 1-8

$$\sum_{n=1}^{6}(n+10)$$

(1 + 10) + (2 + 10) + (3 + 10) + (4 + 10) + (5 + 10) + (6 + 10)

11 + 12 + 13 + 14 + 15 + 16

Examples for 9-30

2 + 4 + 6 + ... + 1000

$$\sum_{n=1}^{n \text{ of } 1000} 2n$$

arithmetic

$$2n = 1000$$

$$\sum_{n=1}^{n \text{ of } 1000} t_1 + (n-1)d$$

$$n = 500$$

$$\sum_{n=1}^{n \text{ of } 1000} 2 + (n-1)2$$

$$\sum_{n=1}^{500} 2n$$

$$\sum_{n=1}^{n \text{ of } 1000} 2 + 2n - 2$$

Examples for 31 & 32

$$\sum_{k=5}^{8} \frac{k}{k+4} = \sum_{j=1}^{4} ?$$

$$\frac{5}{9} + \frac{6}{10} + \frac{7}{11} + \frac{8}{12} = \sum_{j=1}^{4} ?$$

Numerators make an arithmetic series described by the formula j + 4.

Denominators make an arithmetic series described by the formula j + 8.

$$\sum_{j=1}^{4} \frac{j+4}{j+8}$$

Section 11-5

Sums of Arithmetic and Geometric Series

Objectives

- to find the sum of an arithmetic series
- to find the sum of a geometric series
- to solve applications of series

Sums of Finite Arithmetic Series

- The sum of the first n terms of an arithmetic series is

$$S_n = \frac{n(t_1 + t_n)}{2}$$

- S_n is the sum
- t_1 is the first term
- t_n is the last term

Sums of Finite Geometric Series

- The sum of the first n terms of a geometric series with common ratio r is

$$S_n = \frac{t_1\left(1-r^n\right)}{1-r} \qquad (r \neq 1)$$

- S_n is the sum.
- t_1 is the first term
- r is the common ratio
- n is the number of terms in the series

Examples for 1-12

$$\sum_{k=1}^{100} 5k$$

$$t_1 = 5$$

$$t_{100} = 500$$

$$S_n = \frac{n(t_1 + t_n)}{2}$$

$$n = 100$$

$$S_{100} = \frac{100(5 + 500)}{2}$$

$$S_{100} = 50(505)$$

$$S_{100} = 25{,}250$$

Examples for 13-20

$$\sum_{k=1}^{12} 2^{-k}$$

$$t_1 = 2^{-1} = \frac{1}{2}$$

$$t_2 = 2^{-2} = \frac{1}{4}$$

$$r = \frac{\frac{1}{4}}{\frac{1}{2}} = \left(\frac{1}{4}\right)\left(\frac{2}{1}\right) = \frac{1}{2}$$

$$S_n = \frac{t_1\left(1 - r^n\right)}{1 - r}$$

$$S_{12} = \frac{\frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^{12}\right)}{1 - \frac{1}{2}}$$

$$S_{12} = \frac{\frac{1}{2} - \left(\frac{1}{2}\right)^{13}}{\frac{1}{2}}$$

$$S_{12} = \left(\frac{1}{2} - \left(\frac{1}{2}\right)^{13}\right)(2)$$

$$S_{12} = 1 - \left(\frac{1}{2}\right)^{12}$$

$$S_{12} = 1 - \frac{1}{4096}$$

$$S_{12} = \frac{4095}{4096}$$

Examples for 21-28

The positive three digit odd integers.

$$101 + 103 + 105 + ... + 999$$

This is an arithmetic series described by the formula
$t_n = 101 + (n - 1)2 = 2n + 99$

$$S_n = \frac{n(t_1 + t_n)}{2}$$

$$S_n = \frac{n(101 + 999)}{2}$$

$$999 = 2n + 99$$
$$900 = 2n$$
$$n = 450$$

$$S_{450} = \frac{450(101 + 999)}{2}$$

$$S_{450} = \frac{450(1100)}{2}$$

$$S_{450} = 247{,}500$$

Section 11-6

Infinite Geometric Series

Objectives

- to find the sum, if any, of an infinite geometric series
- to approximate the sum of an infinite geometric series
- to write the first three terms of an infinite geometric series
- to solve applications of infinite geometric series

Sums of Infinite Geometric Series

- An infinite geometric series with a common ratio r has a sum S if $|r| < 1$.

$$S = \frac{t_1}{1 - r}$$

- S is the sum
- r is the common ratio
- t_1 is the first term

Examples for 1-16

$$\sum_{n=0}^{\infty} 3\left(\frac{1}{4}\right)^n$$

$$S = \frac{t_1}{1-r}$$

$$3 + \frac{3}{4} + \frac{3}{16} + \ldots$$

$$S = \frac{3}{1-\frac{1}{4}}$$

$$t_1 = 3$$

$$S = \frac{3}{\frac{3}{4}}$$

$$r = \frac{1}{4}$$

$$S = 3\left(\frac{4}{3}\right)$$

$$S = 4$$

Examples for 17-22

3.12312312...

3.12 + 0.00312 + 0.00000312 + ...

$$S = \frac{t_1}{1 - r}$$

$$S = \frac{3.12}{1 - 10^{-3}}$$

$$S = \frac{\frac{312}{100}}{\frac{999}{1000}}$$

$$S = \frac{312}{100}\left(\frac{1000}{999}\right)$$

$$S = \frac{1040}{333}$$

Examples for 23-26

$t_1 = 8$, $S = 12$

$$S = \frac{t_1}{1 - r}$$

$$12 = \frac{8}{1 - r}$$

$12 - 12r = 8$

$4 = 12r$

$$r = \frac{1}{3}$$

$$8 + \frac{8}{3} + \frac{8}{9} + \ldots$$

Section 11-7

Powers of Binomials

Objectives

- to expand a binomial utilizing Pascal’s triangle
- to find terms in the expansion of a binomial by utilizing Pascal’s triangle

Binomial Expansion

$$(a + b)^0 = \mathbf{1}$$

$$(a + b)^1 = \mathbf{1}a + \mathbf{1}b$$

$$(a + b)^2 = \mathbf{1}a^2 + \mathbf{2}ab + \mathbf{1}b^2$$

$$(a + b)^3 = \mathbf{1}a^3 + \mathbf{3}a^2b^1 + \mathbf{3}a^1b^2 + \mathbf{1}b^3$$

$$(a + b)^4 = \mathbf{1}a^4 + \mathbf{4}a^3b^1 + \mathbf{6}a^2b^2 + \mathbf{4}a^1b^3 + \mathbf{1}b^4$$

$$(a + b)^5 = \mathbf{1}a^5 + \mathbf{5}a^4b^1 + \mathbf{10}a^3b^2 + \mathbf{10}a^2b^3 + \mathbf{5}a^1b^4 + \mathbf{1}b^5$$

Pascal's Triangle

	Row
1	**0**
1 1	**1**
1 2 1	**2**
1 3 3 1	**3**
1 4 **6** **4** 1	**4**
1 5 10 **10** 5 1	**5**
1 6 15 20 15 6 1	**6**

Examples for 1-12

$(x^2 - 1)^6$

$1(\ )(\ ) + 6(\ )(\ ) + 15(\ )(\ ) + 20(\ )(\ ) + 15(\ )(\ ) + 6(\ )(\ ) + 1(\ )(\ )$

$1(\ )^6(\ )^0 + 6(\ )^5(\ )^1 + 15(\ )^4(\ )^2 + 20(\ )^3(\ )^3 + 15(\ )^2(\ )^4 + 6(\ )^1(\ )^5 + 1(\ )^0(\ )^6$

$1(x^2)^6(-1)^0 + 6(x^2)^5(-1)^1 + 15(x^2)^4(-1)^2 + 20(x^2)^3(-1)^3 +$

$15(x^2)^2(-1)^4 + 6(x^2)^1(-1)^5 + 1(x^2)^0(-1)^6$

$\mathbf{1x^{12} - 6x^{10} + 15x^8 - 20x^6 + 15x^4 - 6x^2 + 1}$

Examples for 13-16

The first three terms of $(x + y)^{17}$ are $x^{17} + 17x^{16}y + 136x^{15}y^2$.
Write the last three terms.

$136(\ )(\ ) + 17(\ )(\ ) + 1(\ )(\ )$

$136(\ )^2(\ )^{15} + 17(\ )^1(\ )^{16} + 1(\ )^0(\ )^{17}$

$136(x)^2(y)^{15} + 17(x)^1(y)^{16} + 1(x)^0(y)^{17}$

$\mathbf{136x^2y^{15} + 17xy^{16} + y^{17}}$

Examples for 17-22

$(x + y)^7 + (x - y)^7$

The absolute values of the coefficients are the same for both expansions.

The variables and their exponents for each term are the same in both expansions.

The only difference is that every other term in the second expansion will be negative; therefore, every other term will cancel.

This means that you only need to know the 1st, 3rd, 5th, and 7th terms, and there are two of each of these.

$$2[1(\ )(\ ) + 21(\ )(\ ) + 35(\ )(\ ) + 7(\ )(\ )]$$

$$2[1(\ )^7(\ )^0 + 21(\ )^5(\ )^2 + 35(\ )^3(\ )^4 + 7(\ )^1(\ )^6]$$

$$2[1x^7y^0 + 21x^5y^2 + 35x^3y^4 + 7x^1y^6]$$

$$\mathbf{2x^7 + 42x^5y^2 + 70x^3y^4 + 14xy^6}$$

Section 11-8

Binomial Expansion

Objectives

- to evaluate factorials
- to use the binomial expansion theory to expand binomials
- to use the binomial expansion theory to identify a specific term in the expansion of a binomial

Binomial Expansion

- factorial: n! (read n factorial) = (n)(n - 1)(n - 2)(n - 3)...
- 0! = 1

The Binomial Theorem

- If n is a positive integer, then $(a + b)^n$ =

$$a^n + \frac{n!}{(n-1)!1!}a^{n-1}b + \frac{n!}{(n-2)!2!}a^{n-2}b^2 + \frac{n!}{(n-3)!3!}a^{n-3}b^3 + \ldots + b^n$$

  - Notice that the numerator of the coefficient is always the power of the binomial factorial.
  - Notice that the denominator of the coefficient is always the product of the exponents of a factorial and the exponent of b factorial.
  - Notice that the exponent of b is always one less than the position of the term in the polynomial.
  - Notice that the exponents of a and b always have a sum equal to n, the power of the binomial.

Examples for 1-12

$$\frac{(n+1)!}{n!}$$

$$\frac{(n+1)(n)(n-1)(n-2)\ldots}{(n)(n-1)(n-2)\ldots}$$

n +1

Examples for 13-16

$$(a - b)^{14}$$

$$\frac{14!}{(\ )}(a)(-b)+\frac{14!}{(\ )}(a)(-b)+\frac{14!}{(\ )}(a)(-b)+\frac{14!}{(\ )}(a)(-b)$$

$$\frac{14!}{14!0!}(a)^{14}(-b)^{0}+\frac{14!}{13!1!}(a)^{13}(-b)^{1}+\frac{14!}{12!2!}(a)^{12}(-b)^{2}+\frac{14!}{11!3!}(a)^{11}(-b)^{3}$$

$$\frac{14!}{14!0!}a^{14}-\frac{14!}{13!1!}a^{13}b+\frac{14!}{12!2!}a^{12}b^{2}-\frac{14!}{11!3!}a^{11}b^{3}$$

$$1a^{14}-14a^{13}b+\frac{(14)(13)}{2}a^{12}b^{2}-\frac{(14)(13)(12)}{(3)(2)}a^{11}b^{3}$$

$$a^{14}-14a^{13}b+91a^{12}b^{2}-364a^{11}b^{3}$$

Examples for 17-24

The eleventh term of $(s - t)^{14}$

$$\frac{14!}{4!10!} s^4 (-t)^{10}$$

$$\frac{(14)(13)(12)(11)}{(4)(3)(2)} s^4 t^{10}$$

$$1001 s^4 t^{10}$$