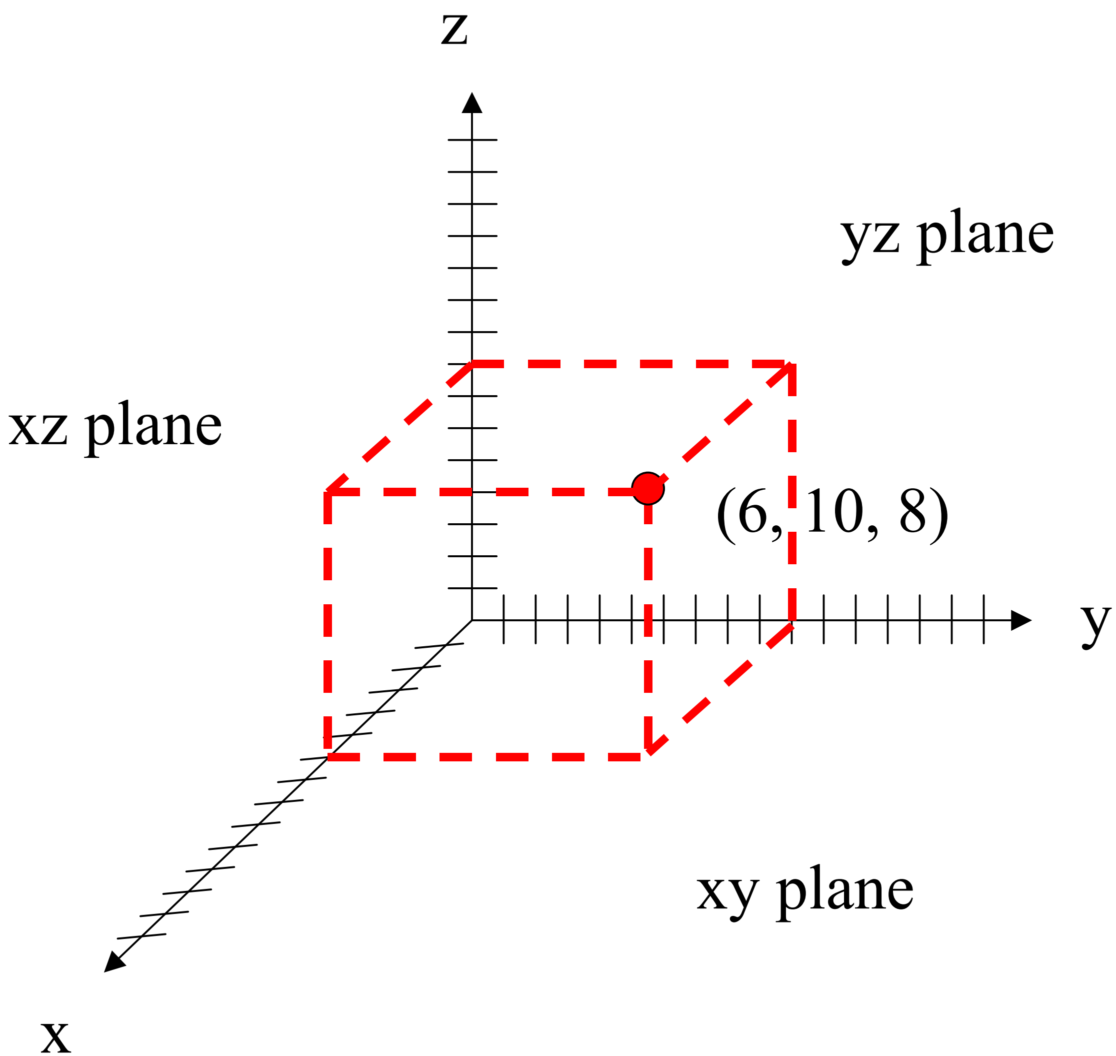
Section 5-1

Coordinates in Space

Objectives for Section 5-1

- to graph a point in three dimensional space
- to sketch a triangle in three dimensional space
- to sketch a plane in three dimensional space

Coordinates in Three Dimensions



Section 5-2

Graphs of Linear Equations in Three Variables

Objectives for Section 5-2

- to graph a linear equation in three variables and to identify its coordinate traces
- to graph a plane given its coordinate traces
- to write a system of equation for a plane when given two of its coordinate traces
- to write the equation of a plane when given two of its coordinate traces

Graphs of Linear Equations in Three Variables

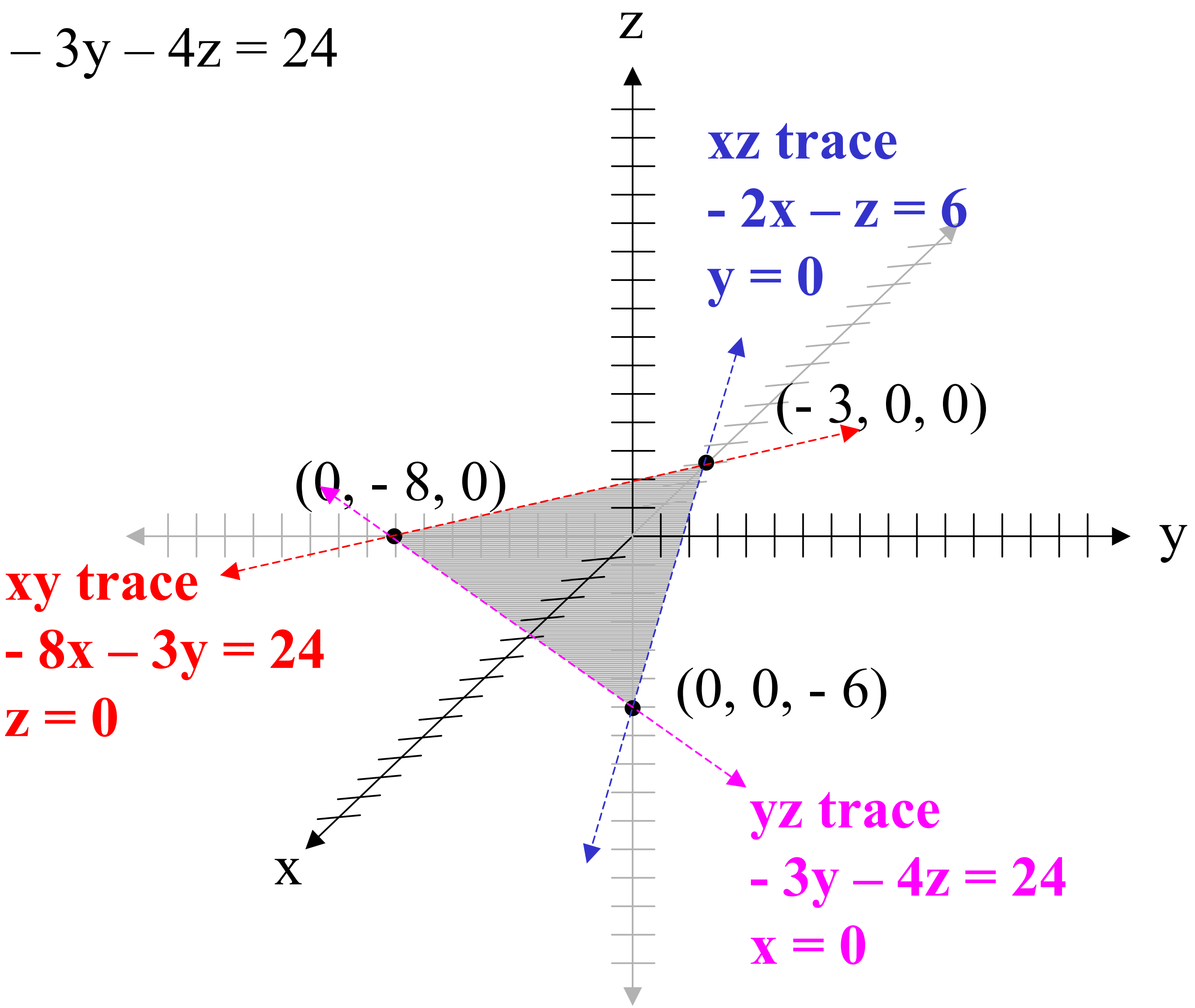
- standard form: $Ax + By + Cz = D$
- In space, the graph of a linear equation in three variables is a plane. Conversely, every plane is the graph of some linear equation in three variables, called an equation of the plane.
- Intercepts of the plane are the points on the axes where the plane crosses.
  - x-intercept of the plane $\left(\frac{D}{A},0,0\right)$
  - y-intercept of the plane $\left(0,\frac{D}{B},0\right)$
  - z-intercept of the plane $\left(0,0,\frac{D}{C}\right)$

Graphs of Linear Equations in Three Variables

- trace: a line in which a plane intersects the coordinate plane
  - slope of the xy trace $-\frac{A}{B}$
  - slope of the yz trace $-\frac{B}{C}$
  - slope of the xz trace $-\frac{A}{C}$
- If the coefficient of a variable in an equation of a plane is zero, then:
  - The plane is parallel to the axis of that variable if the constant term is not zero;
  - the plane contains the axis of that variable if the constant term is zero.

Example for 1-18

$-8x - 3y - 4z = 24$

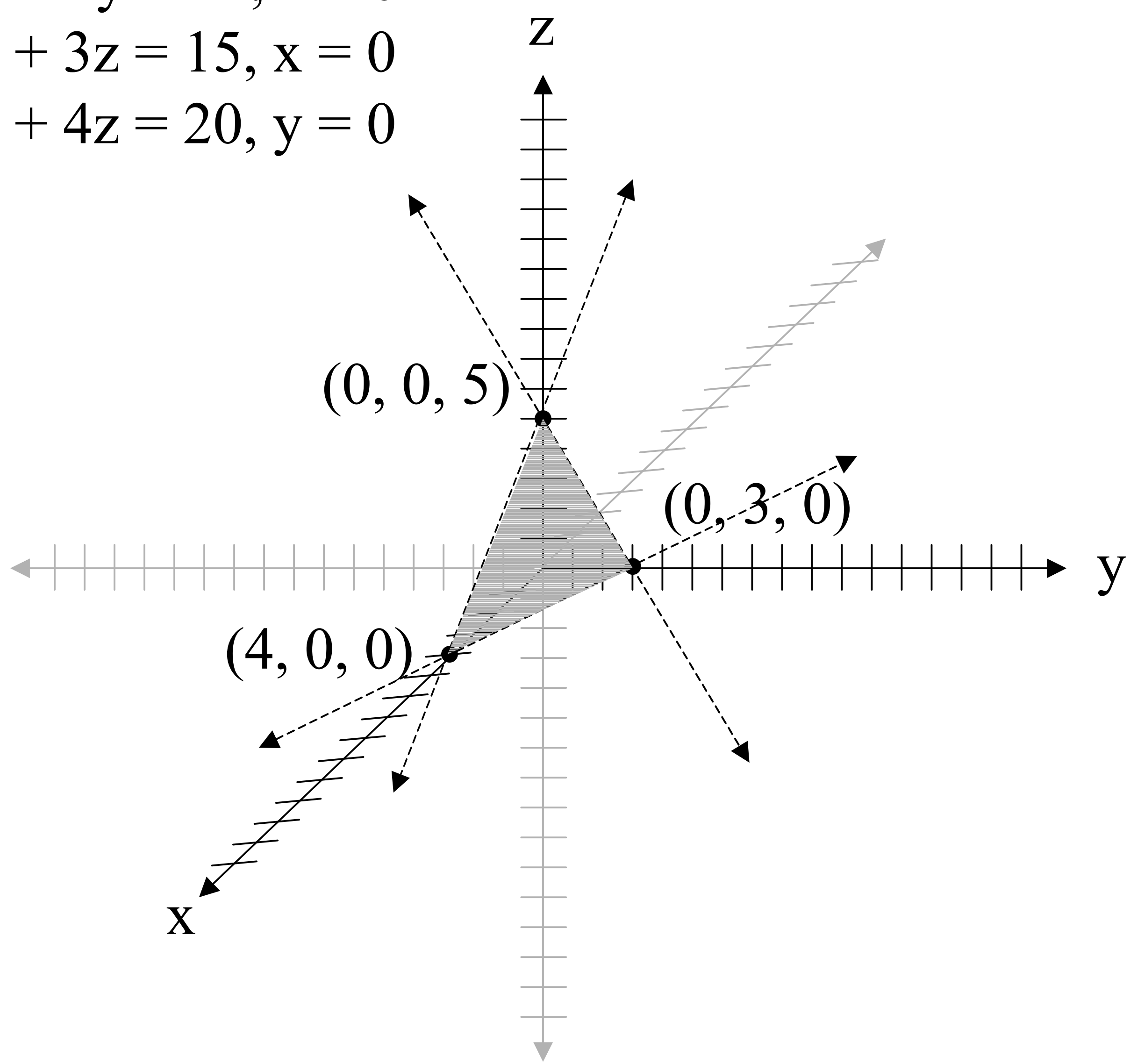


Example for 19-28

xy trace: $3x + 4y = 12$, $z = 0$

yz trace: $5y + 3z = 15$, $x = 0$

xz trace: $5x + 4z = 20$, $y = 0$



Section 5-3

Systems of Linear Equations in Three Variables

Objectives for Section 5-3

- to solve linear systems in three variables
- to determine whether a linear system in three variables is consistent or inconsistent

Systems of Linear Equations in Three Variables

- consistent systems will have either:
  - three planes intersecting in a single point (ordered triple)
  - three different planes intersecting in a single line (infinite solution set)
  - two coincident planes intersecting a third plane in a line (infinite solution set)
  - three coincident planes (infinite solution set)
- inconsistent systems will have either:
  - three parallel planes
  - two coincident planes parallel to a third plane
  - three planes intersecting in three parallel lines
  - two parallel planes intersecting a third in two parallel lines

Example for 1-18

$$\begin{aligned}(I)\ & 2x + 3y = 6 + z \\ (II)\ & x - 2y = -1 - z \\ (III)\ & 3x + y = -1 + 3z\end{aligned}$$

$$\begin{aligned}(I)\ & 2x + 3y - z = 6 \\ (II)\ & x - 2y + z = -1 \\ (III)\ & 3x + y - 3z = -1\end{aligned}$$

$$\begin{aligned}(I)\ & 2x + 3y - z = 6 \\ (II)\ & \underline{x - 2y + z = -1} \\ (IV)\ & 3x + y = 5\end{aligned}$$

$$\begin{aligned}(II)\ & 3[x - 2y + z = -1] \\ (III)\ & \underline{3x + y - 3z = -1}\end{aligned}$$

$$\begin{aligned}(II)\ & 3x - 6y + 3z = -3 \\ (III)\ & \underline{3x + y - 3z = -1} \\ (V)\ & 6x - 5y = -4\end{aligned}$$

$$\begin{aligned}(IV)\ & 5[3x + y = 5] \\ (V)\ & \underline{6x - 5y = -4}\end{aligned}$$

$$\begin{aligned}(IV)\ & 15x + 5y = 25 \\ (V)\ & \underline{6x - 5y = -4} \\ & 21x = 21 \\ & x = 1\end{aligned}$$

$$\begin{aligned}(V)\ & 6(1) - 5y = -4 \\ & -5y = -10 \\ & y = 2\end{aligned}$$

$$\begin{aligned}(I)\ & 2(1) + 3(2) - z = 6 \\ & 2 + 6 - z = 6 \\ & 8 - z = 6 \\ & z = 2\end{aligned}$$

(1, 2, 2)

Example for 1-18

(I) $x - 2y + 3z = 1$
(II) $x + y - 3z = 7$
(III) $3x - 4y + 5z = 7$

(I) $x - 2y + 3z = 1$
(II) $\underline{x + y - 3z = 7}$
(IV) $- 3y + 6z = - 6$

(I) $3[x - 2y + 3z = 1]$
(III) $\underline{3x - 4y + 5z = 7}$

(I) $3x - 6y + 9z = 3$
(III) $\underline{3x - 4y + 5z = 7}$
(V) $- 2y + 4z = - 4$

(IV) $- 3y + 6z = - 6$
(V) $\underline{- 2y + 4z = - 4}$

(IV) $- y + 2z = - 2$
(V) $\underline{- y + 2z = - 2}$

$2z + 2 = y$

(II) $x + (2z + 2) - 3z = 7$

$x = z + 5$

$(z + 5, 2z + 2, z)$

Section 5-4

Third-order Determinants

Objectives for Section 5-4

- to calculate third order determinants
- to use Cramer’s Rule to solve linear systems in three variables
- to use the values of D to determine whether a linear system in three variables is consistent, inconsistent or infinite

Determinant of a Matrix

- Determinant of a 3x3 matrix (3rd order determinant):

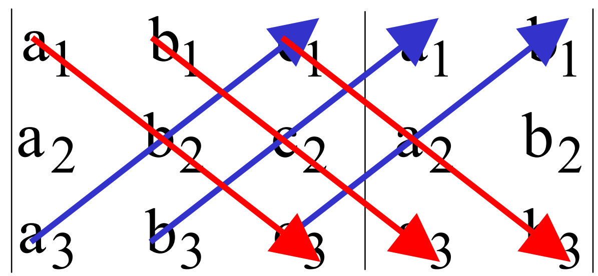
$$\text{Let } B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \text{ then } \det B = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$$

$$a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2$$

Calculating a 3rd Order Determinant

$$\det B = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Copy the first two columns onto the end of the matrix.



Multiply diagonally down through the matrix adding the products.

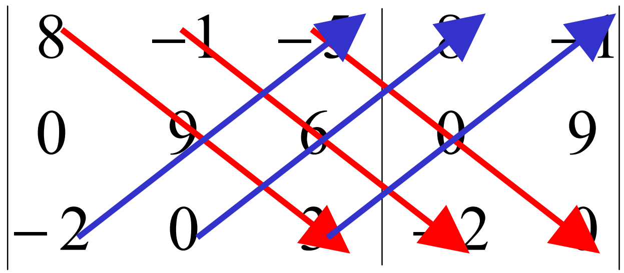
$$a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1$$

Multiply diagonally up through the matrix subtracting the products.

$$a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3$$

Example for 1-4

$$\begin{vmatrix} 8 & -1 & -5 \\ 0 & 9 & 6 \\ -2 & 0 & 3 \end{vmatrix}$$



$$(8)(9)(3) + (-1)(6)(-2) + (-5)(0)(0)$$

$$(8)(9)(3) + (-1)(6)(-2) + (-5)(0)(0) - (-2)(9)(-5) - (0)(6)(8) - (3)(0)(-1)$$

$$(216) + (12) + (0) - (90) - (0) - (0)$$

$$138$$

Example for 5-10

$$2x + y - z = 2$$
$$x + y + z = 7$$
$$x + 2y + z = 4$$

$$D = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} \quad D_x = \begin{vmatrix} 2 & 1 & -1 \\ 7 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix} \quad D_y = \begin{vmatrix} 2 & 2 & -1 \\ 1 & 7 & 1 \\ 1 & 4 & 1 \end{vmatrix} \quad D_z = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 1 & 7 \\ 1 & 2 & 4 \end{vmatrix}$$

$D = -3$ $\quad$ $D_x = -15$ $\quad$ $D_y = 9$ $\quad$ $D_z = -15$

$$x = \frac{-15}{-3} \quad y = \frac{9}{-3} \quad z = \frac{-15}{-3}$$

(5, - 3, 5)

Example for 11-14

$$x - 2y + z = 3$$
$$2x - 4y + 2z = 9$$
$$x = 4$$

$$D = \begin{vmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

slope of the xy trace in first two equations is ½ but the intercepts are different

slope of the yz trace in first two equations is 2 but the intercepts are different

slope of the xz trace in first two equations is – 1 but the intercepts are different

therefore the planes are parallel and intersecting the third plane creating two parallel lines

the system is inconsistent

Section 5-5

Solving Problems with Three Variables

Objectives for Section 5-5

- to solve word problems involving three variables

Solving Problems in Three Variables

- Remember to use the five steps for sorting and organizing the information in the word problem.
- Remember that you must have as many equations as you have variables.
- Once you have set up your system you may use any method you choose to solve it.
- Remember to check to make sure that you have answered the question that you were asked and that this answer makes sense with the question.

Section 5-6

Properties of Determinants

Objectives for Section 5-6

- to use expansion by minors to find the value of fourth order determinants
- to use properties of determinants to simplify the process of expansion by minors
- to use Cramer's Rule to solve a linear system in four variables

Expansion of Determinants by Minors

- minor of an element: is the determinant resulting from the deletion of one row and one column containing the element
- this process can be used on determinants of the 3rd or higher order
- expanding by minors:
  - multiply each element of a given row or column by its minor
  - add the numbers of the index of the elements, if the sum is odd subtract the product from the determinant, if the sum is even add the product to the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Properties of Determinants

- Property 1: If each element in any row or column is 0, then the determinant is equal to 0.
- Property 2: If two rows or columns of a determinant have corresponding elements that are equal, then the determinant is 0.
- Property 3: If two rows or columns of a determinant are interchanged, then the resulting determinant is the opposite of the original determinant.
- Property 4: If each element in one row or column of a determinant is multiplied by a real number k, then the determinant is multiplied by k.
- Property 5: If each element of one row or column is multiplied by a real number k and if the resulting products are then added to the corresponding elements of another row or column, then the resulting determinant equals the original one.

Example for 1-3

$$\begin{vmatrix} 2 & 1 & 3 \\ -2 & 1 & 4 \\ 1 & 2 & 5 \end{vmatrix}$$ row one

$$2\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} - 1\begin{vmatrix} -2 & 4 \\ 1 & 5 \end{vmatrix} + 3\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix}$$

2(5 - 8) - 1(- 10 - 4) + 3(- 4 - 1)

2(- 3) - 1(- 14) + 3(- 5)

- 6 + 14 - 15

- 7

Example for 4-6

$$\begin{vmatrix} 5 & 0 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ 8 & 2 & 2 & 0 \\ -4 & 10 & 3 & 2 \end{vmatrix}$$

Expand the 4th column to take advantage of the 3 zeroes.

$$-0\begin{vmatrix} 3 & -3 & 0 \\ 8 & 2 & 2 \\ -4 & 10 & 3 \end{vmatrix} + 0\begin{vmatrix} 5 & 0 & 0 \\ 8 & 2 & 2 \\ -4 & 10 & 3 \end{vmatrix} - 0\begin{vmatrix} 5 & 0 & 0 \\ 3 & -3 & 0 \\ -4 & 10 & 3 \end{vmatrix} + 2\begin{vmatrix} 5 & 0 & 0 \\ 3 & -3 & 0 \\ 8 & 2 & 2 \end{vmatrix}$$

$$2\begin{vmatrix} 5 & 0 & 0 \\ 3 & -3 & 0 \\ 8 & 2 & 2 \end{vmatrix}$$

Expand the 3rd column to take advantage of the 2 zeroes.

$$2\left(0\begin{vmatrix} 3 & -3 \\ 8 & 2 \end{vmatrix} - 0\begin{vmatrix} 5 & 0 \\ 8 & 2 \end{vmatrix} + 2\begin{vmatrix} 5 & 0 \\ 3 & -3 \end{vmatrix}\right)$$

$$2[2(-15-0)]$$

- 60

Example for 7&8

2a - 3b + c - 3d = - 1

a + 2b + 3c - d = 1

3a + 5b + 6c = 4

3a - b - 2d = 6

$$D_b = \begin{vmatrix} 2 & -1 & 1 & -3 \\ 1 & 1 & 3 & -1 \\ 3 & 4 & 6 & 0 \\ 3 & 6 & 0 & -2 \end{vmatrix} = 112$$

$$D = \begin{vmatrix} 2 & -3 & 1 & -3 \\ 1 & 2 & 3 & -1 \\ 3 & 5 & 6 & 0 \\ 3 & -1 & 0 & -2 \end{vmatrix} = 56$$

$$D_c = \begin{vmatrix} 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & -1 \\ 3 & 5 & 4 & 0 \\ 3 & -1 & 6 & -2 \end{vmatrix} = -112$$

$$a = \frac{112}{56}$$

$$b = \frac{112}{56}$$

$$c = \frac{-112}{56}$$

$$d = \frac{-56}{56}$$

$$D_a = \begin{vmatrix} -1 & -3 & 1 & -3 \\ 1 & 2 & 3 & -1 \\ 4 & 5 & 6 & 0 \\ 6 & -1 & 0 & -2 \end{vmatrix} = 112$$

$$D_d = \begin{vmatrix} 2 & -3 & 1 & -1 \\ 1 & 2 & 3 & 1 \\ 3 & 5 & 6 & 4 \\ 3 & -1 & 0 & 6 \end{vmatrix} = -56$$

(2, 2, - 2, - 1)