Section 6-1

Laws of Exponents

Objectives for Section 6-1

- to write expressions using only positive exponents
- to prove theorems concerning exponents

Laws of Exponents

- If a and b $\in \Re$, and m and n are positive integers,
  - then $b^m b^n = b^{m+n}$
  - then $(b^m)^n = b^{mn}$
  - then $(ab)^m = a^m b^m$
  - then if m > n and $b \neq 0$, $\dfrac{b^m}{b^n} = b^{m-n}$
  - then if m < n and $b \neq 0$, $\dfrac{b^m}{b^n} = \dfrac{1}{b^{n-m}}$
  - then if $b \neq 0$, $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$
  - then if $b \notin \{-1, 0, 1\}$, then $b^m = b^n$ if and only if $m = n$

Laws of Exponents

- For all r, s, t, and u $\in \Re$, and t, u $\neq 0$,
  - $\frac{rs}{tu} = \frac{r}{t} \bullet \frac{s}{u}$
  - $\frac{rs}{u} = r \bullet \frac{s}{u}$
  - $\frac{s}{tu} = \frac{1}{t} \bullet \frac{s}{u}$
- For all nonzero b $\in \Re$ and all positive integers n,
  - $b^0 = 1$
  - $b^{-n} = \frac{1}{b^n}$

Example for 1-26

$$\frac{a^{-1} - b^{-1}}{a^{-1} + b^{-1}}$$

$$\frac{\frac{1}{a} - \frac{1}{b}}{\frac{1}{a} + \frac{1}{b}}$$

$$\frac{b - a}{b + a}$$

Section 6-2

Multiplying Polynomials

Objectives for Section 6-2

- to find the product of polynomials in their simplest form

Multiplying Polynomials

- When asked to multiply polynomials you must multiply each term of the first factor to each term of the second factor.
- If there are more than two factors, then you may only work with two at a time.
- When both factors are binomials you can simplify the distribution process by using the FOIL (firsts-outsides-insides-lasts) method
- Common FOIL patterns are:
  - $(a + b)^2 = a^2 + 2ab + b^2$
  - $(a - b)^2 = a^2 - 2ab + b^2$
  - $(a + b)(a - b) = a^2 - b^2$

Example for 1-28

Distribute every term in the first factor into the second factor.
For example # 31, $(x^2 - x + 2)(x^2 + x - 1)$

$$x^2(x^2) + x^2(x) + x^2(-1) + (-x)(x^2) + (-x)(x) + (-x)(-1) + 2(x^2) + 2(x) + 2(-1)$$

$$x^4 + x^3 - x^2 - x^3 - x^2 + x + 2x^2 + 2x - 2$$

$$x^4 + 3x - 2$$

Example for 31-36

$$\left(r^{4n} - s^{3m}\right)^2$$

$$\left(r^{4n} - s^{3m}\right)\left(r^{4n} - s^{3m}\right)$$

$$\left(r^{4n}\right)\left(r^{4n}\right) + \left(r^{4n}\right)\left(-s^{3m}\right) + \left(-s^{3m}\right)\left(r^{4n}\right) + \left(-s^{3m}\right)\left(-s^{3m}\right)$$

$$r^{8n} - 2r^{4n}s^{3m} + s^{6m}$$

Section 6-3

Factoring a Polynomial

Objectives for Section 6-3

- to factor polynomials completely and to identify prime polynomials

Factoring Polynomials

I. Factor out the GCF if any exists.

II. Look for one of the 5 patterns of factoring.

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$

$(a + b)(a - b) = a^2 - b^2$

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

III. Arrange the terms in decreasing order by degree to see if they fit the standard form of a quadratic equation: $ax^2 + bx + c$

IV. Standard form quadratic equations are created by FOILing binomials; therefore, ax^2 comes from the product of Firsts, c comes from the product of Lasts & bx comes from the sum of the Outside and Inside products.

Factoring Polynomials

V. Set up two sets of parentheses.

VI. In the Firsts position write the factors that will have the product ax^2.

VII. Multiply (a)(c) and find a factor pair of that product with a sum of b. The numbers in the factor pair represent either your Outside or Inside products.

VIII. Insert the necessary values into the Lasts positions so that the product of your Outsides and Insides are the numbers in your factor pair.

IX. Check your answer by reFOILing. You should get the original polynomial back.

Example for 1-34

I. $3p^2 - 7p - 6$

II. $(ac) = (3)(-6) = -18$

factor pairs: $\{1, -18\}$ $\{-1, 18\}$ $\{2, -9\}$ $\{-2, 9\}$ $\{3, -6\}$ $\{-3, 6\}$

$\{2, -9\}$ is the factor pair we want to use because $2 + (-9) = -7$

III. $(3p \quad\quad)(p \quad\quad)$

IV. $(3p + 2)(p - 3)$

V. $(3p)(p) + (3p)(-3) + (2)(p) + (2)(-3)$

$3p^2 - 9p + 2p - 6$

$3p^2 - 7p - 6$

Example for 1-34

$$(r+1)^3 - s^3$$

$$[(r+1)-s]\left[(r+1)^2 + (r+1)(s) + s^2\right]$$

$$[r+1-s][r^2 + 2r + 1 + rs + s + s^2]$$

Section 6-4

Solving Equations by Factoring

Objectives for Section 6-4

- to solve polynomial equations by factoring
- to solve word problems involving factorable polynomials

Solving Polynomial Equations

- a polynomial equation describes a curved line
- the roots of a polynomial equation are the numbers where the graph of the curve crosses the x-axis; therefore, they are the values of the variable that make the value of the polynomial equal to 0.
- Steps for solving:
  - move all terms to one side--the side where the quadratic term will be positive. If you are moving variables you may have to expand and then re-factor.
  - factor the polynomial side of the equation
  - set each factor equal to 0 and solve for the variable

Problem Solving with Polynomial Equations

- all of the problems in this section require that you set up a quadratic equation that is factorable.
- most of the equations will have you establish the quadratic by substituting into a product of variables.
- make sure that the answers you provide make sense with what the word problem asks.   Not all algebraic solutions apply in the real world.
- make sure you read through the section because it has formulas which you need for some of the problems

Problem Solving with Polynomial Equations

- Make sure that you use the 5 step process to solve these problems.
- Make sure that you are translating correctly by looking back at your chapter 1 notes.
- Make sure you are using the correct formulas by reading the section or recalling them from Geometry.

Example for 1-25

I. $y^2 - 8y + 16 = 2y$

II. $y^2 - 8y - 2y + 16 = 0$

III. $y^2 - 10y + 16 = 0$

IV. $(y - 2)(y - 8) = 0$

V. Either $y - 2 = 0$ or $y - 8 = 0$; therefore, $y = 2$ or 8

Section 6-5

Solving Inequalities by Factoring

Objectives for Section 6-5

- to solve and graph the solution sets to polynomial inequalities

Solving Polynomial Inequalities

- move every thing to one side of the inequality and factor as you would a polynomial equation.
- if the polynomial is greater than 0, then find the combinations of (+) and (-) factors that will make a positive product.
- if the polynomial is less than 0, then find the combinations of (+) and (-) factors that will make a negative product.
- to say that a factor is negative is to say that it is less than 0
- to say that a factor is positive is to say that it is greater than 0

Example for Quadratic Inequalities $t^2 > 9(t - 2)$

I. $t^2 - 9t + 18 > 0$ which factors to $(t - 6)(t - 3) > 0$

II. the product is positive

III. (+)(+) or (-)(-)

IV. $t - 6 > 0$ and $t - 3 > 0$ OR $t - 6 < 0$ and $t - 3 < 0$

$t > 6$ and $t > 3$ OR $t < 6$ and $t < 3$

V.

Example for Inequalities of 3rd Degree or More, $x^3 - 16x > 0$

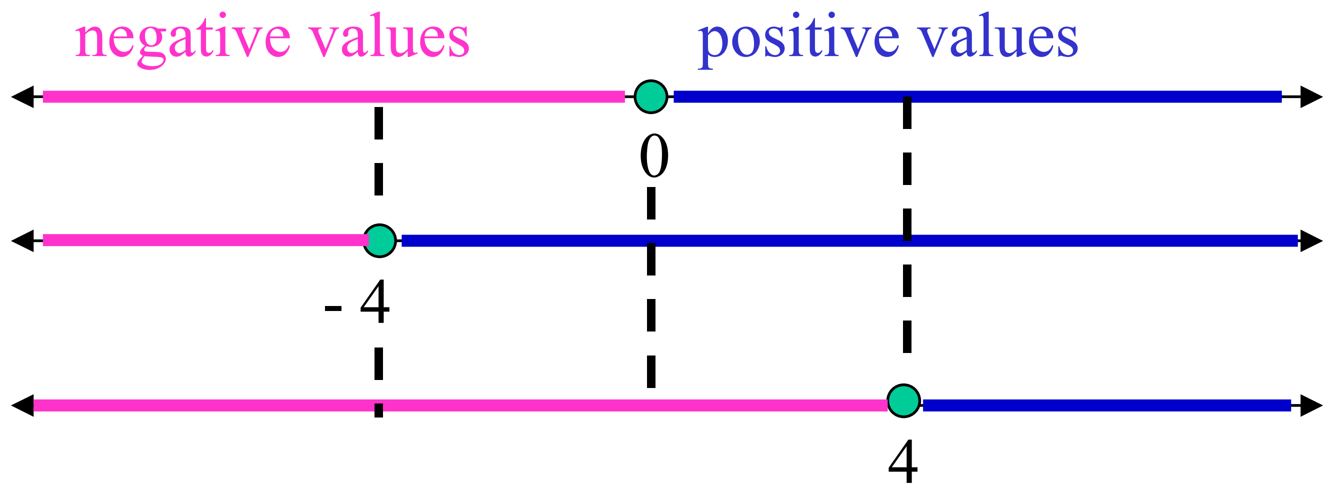
I. $x^3 - 16x > 0$

II. $x(x + 4)(x - 4)$

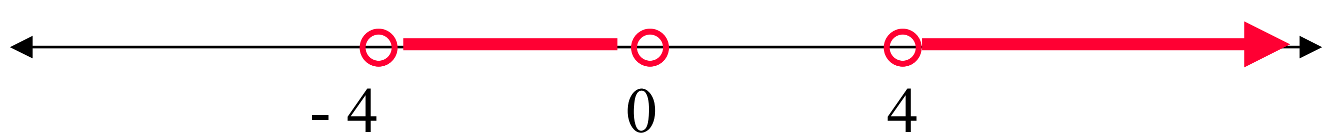
III. x

$x + 4$

$x - 4$



IV. The product is positive so that answers will only be in regions where all three number lines indicate positive values or two negative values.



Section 6-6

Simplifying Rational Expressions

Objectives for Section 6-6

- to simplify rational polynomial expressions

Rational Algebraic Expressions

- rational algebraic expression is a polynomial fraction
- simplifying rational expressions: remember you are using the same process to do these problems as you use to reduce numerical fractions
  - factor the top and the bottom into prime factors
  - cancel any factors which appear in both the numerator and the denominator.
  - negative exponents indicate that the value should be on the other side of the fraction bar.
    - If it is in the numerator, then put it in the denominator.
    - If it is in the denominator then put it in the numerator.

Example for 1-28

$$\frac{x^2 - 5x + 6}{x^2 - 7x + 12}$$

$$\frac{x^2 - 5x + 6}{x^2 - 7x + 12} = \frac{(x-2)(x-3)}{(x-4)(x-3)} = \frac{x-2}{x-4}$$

Section 6-7

Dividing One Polynomial by Another

Objectives for Section 6-7

- to perform polynomial long division

Polynomial Long Division

- polynomial long division works the same as numerical long division
- $$\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}$$
- $\text{Dividend} = (\text{Quotient})(\text{Divisor}) + \text{Remainder}$

Steps to perform polynomial long division

- Write both the dividend and the divisor in decreasing order by degree.
- If any powers of the variable are missing (in either the dividend or the divisor), then insert a term raised to that power with a coefficient of zero.
- Determine what factor must be multiplied by the first term in the divisor to make it equal the first term in the dividend. That factor will be the first term in the quotient.
- Multiply the term to all of the terms in the divisor and subtract from the dividend. Repeat this process until you can divide no longer.

Example of polynomial long division

$$\frac{x^4 - x^3 - 10x + 10}{x - 3} \qquad x-3\overline{\smash{)}\,x^4 - x^3 + 0x^2 - 10x + 10}$$

$$
\begin{array}{r}
x^3 + 2x^2 + 6x + 8 \\
x-3\overline{\smash{)}\,x^4 - x^3 + 0x^2 - 10x + 10} \\
\underline{x^4 - 3x^3} \qquad\qquad\qquad\qquad \\
2x^3 + 0x^2 \qquad\qquad\quad \\
\underline{2x^3 - 6x^2} \qquad\qquad\quad \\
6x^2 - 10x \qquad \\
\underline{6x^2 - 18x} \qquad \\
8x + 10 \\
\underline{8x - 24} \\
34
\end{array}
$$

$$x^3 + 2x^2 + 6x + 8 + \frac{34}{x-3}$$

Example of polynomial long division

$$\frac{9x^4+6x^3+4x+4}{3x^2+2x+2}$$

$$3x^2+2x+2\,\overline{\big)\,9x^4+6x^3+0x^2+4x+4}$$

$$
\begin{array}{r}
3x^2 - 2 \\
3x^2+2x+2\,\overline{\big)\,9x^4+6x^3+0x^2+4x+4} \\
\underline{9x^4+6x^3+6x^2} \\
-6x^2+4x+4 \\
\underline{-6x^2-4x-4} \\
8x+8
\end{array}
$$

$$3x^2-2+\frac{8x+8}{3x^2+2x+2}$$

Example of polynomial long division with two variables

$$\frac{x^6 - a^6}{x^2 + ax + a^2}$$

$$
\begin{array}{r}
x^4 - ax^3 + a^3x - a^4 \\
x^2 + ax + a^2 \,\big)\, x^6 + 0ax^5 + 0a^2x^4 + 0a^3x^3 + 0a^4x^2 + 0a^5x - a^6 \\
\underline{x^6 + ax^5 + a^2x^4} \qquad\qquad\qquad\qquad\qquad\qquad \\
- ax^5 - a^2x^4 + 0a^3x^3 \qquad\qquad\qquad\quad \\
\underline{- ax^5 - a^2x^4 - a^3x^3} \qquad\qquad\qquad\quad \\
a^3x^3 + 0a^4x^2 + 0a^5x \qquad \\
\underline{a^3x^3 + a^4x^2 + a^5x} \qquad \\
- a^4x^2 - a^5x - a^6 \\
\underline{- a^4x^2 - a^5x - a^6} \\
0
\end{array}
$$

$\mathbf{x^4 - ax^3 + a^3x - a^4}$

Section 6-8

Multiplying and Dividing Rational Expressions

Objectives for Section 6-8

- to simplify products and quotients of rational polynomial expressions

Products & Quotients of Rational Expressions

- Remember that division of a fraction is the same as multiplying by the reciprocal.

$$\frac{3}{5} \div \frac{6}{10} = \frac{3}{5} \bullet \frac{10}{6} = 1 \text{ or } \frac{\frac{3}{5}}{\frac{6}{10}} = \frac{3}{5} \bullet \frac{10}{6} = 1$$

Example for 1-24

$$\frac{3x^2 + xy - 2y^2}{3x^2 - xy - 2y^2} \div \frac{3x^2 + 7xy - 6y^2}{3x^2 - 2xy - y^2} \div \frac{3x + y}{3x + 2y}$$

Turn division into multiplication by reciprocals.

$$\frac{3x^2 + xy - 2y^2}{3x^2 - xy - 2y^2} \bullet \frac{3x^2 - 2xy - y^2}{3x^2 + 7xy - 6y^2} \bullet \frac{3x + 2y}{3x + y}$$

Factor

$$\frac{(3x - 2y)(x + y)}{(3x + 2y)(x - y)} \bullet \frac{(3x + y)(x - y)}{(3x - 2y)(x + 3y)} \bullet \frac{3x + 2y}{3x + y}$$

Cancel

$$\frac{\cancel{(3x - 2y)}(x + y)}{\cancel{(3x + 2y)}\cancel{(x - y)}} \bullet \frac{\cancel{(3x + y)}\cancel{(x - y)}}{\cancel{(3x - 2y)}(x + 3y)} \bullet \frac{\cancel{3x + 2y}}{\cancel{3x + y}}$$

Answer: $\frac{x + y}{x + 3y}$

Section 6-9

Adding and Subtracting Rational Expressions

Objectives for Section 6-9

- to find the sums and differences of rational polynomial expressions

Sums & Differences of Rational Expressions

- Follow the same procedure for adding and subtracting polynomial fractions as you would for adding and subtracting numerical fractions.
- Rules for adding and Subtracting Fractions
  - Find the least common denominator (LCM of denominators) of all the fractions.
  - Multiply the numerator and the denominator by the factor of the LCM not present in the denominator already.
  - Add or subtract the fractions by combining similar terms in the numerators and placing this over the LCD.
  - Reduce the sum or difference by factoring and canceling, if possible.

Example for 1-23

$$\frac{1}{2pq^4} + \frac{2}{p^3q^2}$$

The LCM of $2pq^4$ & p^3q^3 is $2p^3q^4$. The first term must be multiplied by p^2 on top and bottom and the second term must be multiplied by $2q^2$ on the top and bottom.

$$\left(\frac{p^2}{p^2}\right)\left(\frac{1}{2pq^4}\right) + \left(\frac{2q^2}{2q^2}\right)\left(\frac{2}{p^3q^2}\right)$$

Multiplying creates the new sum.

$$\left(\frac{p^2}{2p^3q^4}\right) + \left(\frac{4q^2}{2p^3q^4}\right)$$

Adding creates the fraction:

$$\frac{p^2 + 4q^2}{2p^3q^4}$$

Example for 1-23

$$\frac{1}{s^2+2s+1}-\frac{1}{s^2-1}$$

After factoring the denominators, find the LCM of the denominators which in this case is (s + 1)(s + 1)(s - 1).

$$\frac{1}{(s+1)(s+1)}-\frac{1}{(s+1)(s-1)}$$

Multiply the numerator & denominator of the first term by (s - 1) and the second term by (s + 1).

$$\left[\frac{(s-1)}{(s-1)}\right]\left[\frac{1}{(s+1)(s+1)}\right]-\left[\frac{(s+1)}{(s+1)}\right]\left[\frac{1}{(s+1)(s-1)}\right]$$

Problem Continued

The new difference is: $$\left[\frac{s-1}{(s+1)(s+1)(s-1)}\right]-\left[\frac{s+1}{(s+1)(s+1)(s-1)}\right]$$

Combining these terms you get: $$\frac{s-1-(s+1)}{(s+1)(s+1)(s-1)}$$

Distributing the negative: $$\frac{s-1-s-1}{(s+1)(s+1)(s-1)}$$

Simplifying the numerator you get: $$\frac{-2}{(s+1)(s+1)(s-1)}$$

Section 6-10

Using Polynomials with Rational Coefficients

Objectives for Section 6-10

- to solve word problems involving polynomials with rational coefficients

Solving rational open sentences.

I. Find the LCD of all of the fractions.

II. Multiply every term in the problem by the LCD.

III. Reduce each term in the problem and if you've used the correct LCD then no fractions should remain.

IV. Solve the remaining problem according to the appropriate methods you've learned throughout the year.

Example for 1-14

$$\frac{x(x+1)}{5} - \frac{x+1}{6} = \frac{1}{3}$$

The LCD is 30. Multiplying every term by 30 you get:

$$(30)\left(\frac{x(x+1)}{5}\right) - (30)\left(\frac{x+1}{6}\right) = (30)\left(\frac{1}{3}\right)$$

Which simplifies to:

$$6x(x+1) - 5(x+1) = 10$$

Distributing you get: $6x^2 + 6x - 5x - 5 = 10$

Collecting like terms you get: $6x^2 + x - 15 = 0$

Factoring you get: $(3x+5)(2x-3) = 0$

Solution is: $x = \left\{-\frac{5}{3}, \frac{3}{2}\right\}$

Section 6-11

Fractional Equations

Objectives for Section 6-11

- to solve fractional equations
- to solve word problems involving fractional equations

Solving rational open sentences.

I. Identify the LCD.

II. Identify the domain so that you know which values to exclude from the solution set.

III. Multiply every term in the open sentence by the LCD and reduce so that no fractions remain.

IV. Solve the resulting problem in the appropriate fashion.

V. Check all of the answers in the solution set to identify any extraneous roots (values excluded in the domain).

Example for 1-24

$$\frac{5}{u^2+u-6} = 2 - \frac{u-3}{u-2}$$

$$\frac{5}{(u+3)(u-2)} = 2 - \frac{u-3}{u-2}$$

LCD = (u - 2)(u + 3)

Domain = {u: u ≠ 2 or - 3}

$$5 = 2(u+3)(u-2) - (u+3)(u-3)$$

$5 = 2(u^2 + u - 6) - (u^2 - 9)$

$5 = 2u^2 + 2u - 12 - u^2 + 9$

$5 = u^2 + 2u - 3$

$0 = u^2 + 2u - 8$

$0 = (u + 4)(u - 2)$

u = - 4 or 2 but 2 is extraneous

so - 4 is the answer