Section 7-1

Power Functions and Variation

Objectives for Section 7-1

- to graph a pair of simple quadratic equations
- to find the value of a constant so that points lie on the graph of a simple quadratic equation
- to solve problems involving direct variation
- to determine how the value of a simple quadratic function changes given changes in its equation

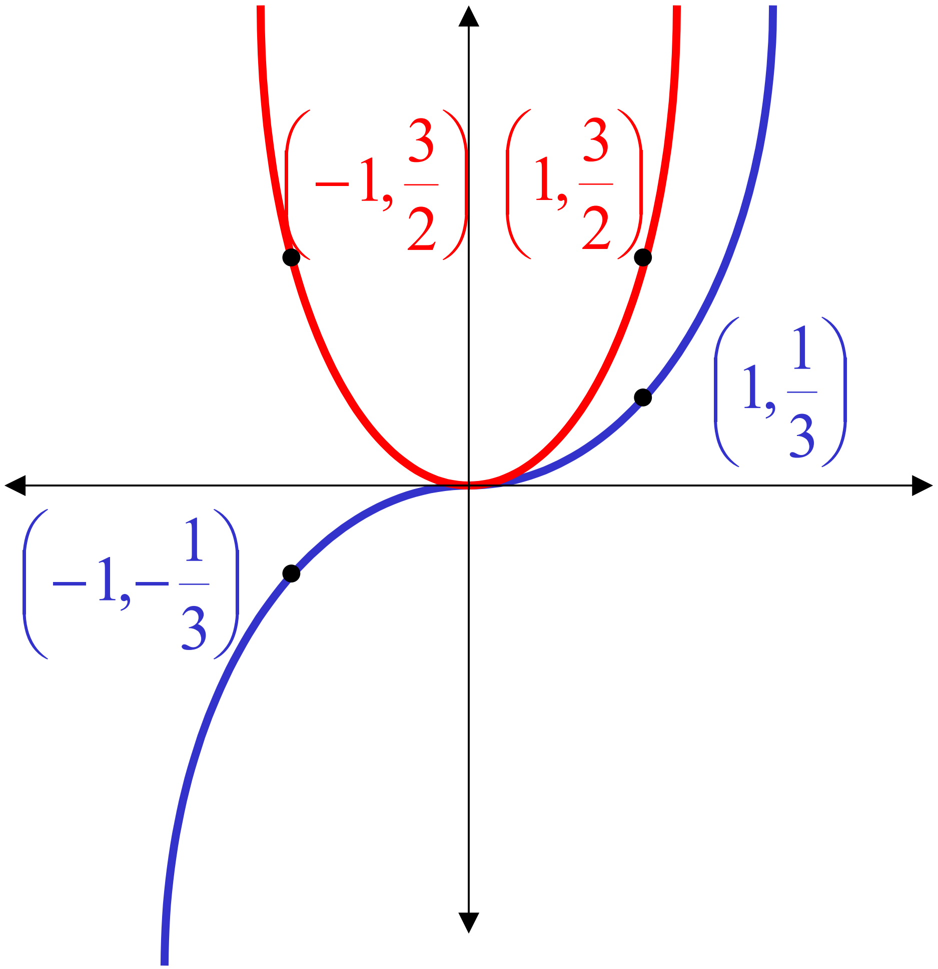
Power Functions and Variation

- $f(x) = x^n$ is a power function
- odd function: an equation whose graph is symmetric with respect to the origin. The graph must contain the points (a, b) and (- a, - b).
- even function: an equation whose graph is symmetric with respect to vertical axis. The graph must contain the points (a, b) and (- a, b).
- $y = ax^n$ where $n > 0$, $a \neq 0$ is a variation. It is read "y varies directly as x" or "y is directly proportional to x." "a" is the constant of variation or the constant of proportionality.

Example for 1-6

$$y = \frac{3}{2}x^2$$

$$y = \frac{1}{3}x^3$$



Example for 7-12

(2, 48) $y = kx^3$

$48 = k(2^3)$

$48 = 8k$

$k = 6$

Example for 13-16

y varies directly as x^2

If y is 36 when x is 9, find x when y is 24.

$$\frac{36}{9^2} = \frac{24}{x^2}$$

$$x^2 = \frac{(24)(9)(9)}{36}$$

$$x^2 = (6)(9)$$

$$x = \sqrt{(9)(6)} = 3\sqrt{6}$$

Example for 17-22

$f(x) = 4x^2$, x is tripled

$$f(3x) = 4(3x)^2$$

$$f(3x) = 4(9)(x^2)$$

$$\mathbf{f(3x) = (9)(4x^2) = 9(f(x))}$$

Section 7-2

The Real n^{th} Roots of a Number

Objectives for Section 7-2

- to solve quadratic equations over the set of rational numbers
- to simplify radical statements
- to find the values which make a radical statement true

The Real nth Roots of a Number

- The graphs of power functions can tell us about the nature of the nth roots of a number. The solution to the equation $x^n = b$ where n is positive integer is called an nth root of b.
- Number and Nature of Real nth Roots of b

	$b > 0$	$b < 0$	$b = 0$
n is even	one positive & one negative root	no real roots	one root = 0
n is odd	one positive root	one negative root	one root = 0

The nth Root of b

- $\sqrt[n]{b}$ read the “nth root of b” denotes the principal nth root of b which is the nonnegative nth root of b if n is even and $b \geq 0$.
  - You use only the principal root when simplifying an expression but you must use both the principal root and the secondary root when solving.
- b is the radicand and n is the index. When no index is written it is assumed to be 2

Example for 1-12

$x^2 - 25 = 0$ could be solved by factoring and setting each factor equal to zero like we did in chapter 4. You can also isolate the x and solve for its value.

$$x^2 = 25$$

$$\sqrt{x^2} = \sqrt{25}$$

$$x = \pm\sqrt{25} = \pm 5$$

Example for 13-16

$$\sqrt[5]{-32} - \sqrt{(-5)^2}$$

$$-2 - \sqrt{25}$$

$$-2 - 5 = -7$$

Example for 17-28

$$\sqrt{x^2} = -x$$

We know that they left side of the equation will always simplify to be a positive number; therefore, we need to find the values of x where $-x \geq 0$. Solving that inequality we know that $\mathbf{x \leq 0}$.

Section 7-3

Roots of Polynomial Equations

Objectives for Section 7-3

- to find the rational roots of an equation
- to write an equation with integral coefficients given one root
- to demonstrate that a number is irrational

Roots of a Polynomial Equation

- Rational Root Theorem: Let f(x) be a simplified polynomial with integral coefficients. If the equation f(x) = 0 has a rational root $\frac{p}{q}$ that is in lowest terms, then p must be an integral factor of the constant term of f(x) and q must be an integral factor of the leading coefficient of f(x).
- With the exceptions of (0)(x) and $\frac{0}{x}$, where x is irrational, the sum, difference, product, or quotient of a rational number and an irrational number is an irrational number.

Example for 1-10

$$2x^3 - 9x^2 + 10x - 3$$

If any of the answers to this problem are going to be rational, then the numerator of those rational numbers must be a factor of (– 3) and the denominator of that rational number must be a factor of (2). Any rational values of x must therefore come from the possibilities of the fraction:

$$\frac{\pm 1 \pm 3}{\pm 1 \pm 2}$$

possible rational values for x are:

$$\left\{ \pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2} \right\}$$

Example for 11-16

$$6x^3 + 7x^2 - 9x + 2 = 0$$

According to the rational root theorem
the possible rational roots are:

$$\frac{\pm 1 \pm 2}{\pm 1 \pm 2 \pm 3 \pm 6} = \left\{ \pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{1}{6} \right\}$$

We know that the equation has a degree of 3 so that there are exactly 3 roots, not all of which must be rational. Testing these possible rational roots in the equation we find that the rational solutions are:

$$x \in \left\{ -2, \frac{1}{2}, \frac{1}{3} \right\}$$

Example for 17-22

$$\sqrt[3]{2}$$

$$x = \sqrt[3]{2}$$

$$x^3 = 2$$

$$x^3 - 2 = 0$$

According to the rational root theorem the only possible rational roots are:

$$\frac{\pm 1 \pm 2}{\pm 1} = \{\pm 1 \pm 2\}$$

Since none of these values work in the equation it has not rational roots; therefore, $\sqrt[3]{2}$ **cannot be a rational number.**

Example for 23-28

If $6-\sqrt[3]{2}$ is a rational number, there must be integers a and b, $b \neq 0$, such that

$$6-\sqrt[3]{2}=\frac{a}{b} \quad \text{or} \quad \sqrt[3]{2}=\frac{a-6b}{-b}=\frac{6b-a}{b}$$

Since both 6b – a and b are integers and $b \neq 0$, $\frac{6b-a}{b}$ is a rational number. The previous problem shows that $\sqrt[3]{2}$ is an irrational number. Thus there is a contradiction and the hypothesis that $6-\sqrt[3]{2}$ is a rational number must be false. Hence, $6-\sqrt[3]{2}$ is irrational.

Section 7-4

Standard Notation

Objectives for Section 7-4

- to express decimal numbers in standard notation
- to express standard notation numbers in decimal form
- to find a significant digit estimate of a product/quotient of standard numbers

Standard Notation

- rounding error: when rounding by the rule of five the difference between a number and its approximation is at most half the unit of the last digit retained.
- precision: is given by the unit used in making the measurement
- maximum possible error: is half the unit of precision
- accuracy: is the relative error, usually expressed as the ratio of the maximum possible error in the measurement to the measurement itself
- significant digit: each digit reporting the number of units of measure contained in a measurement

To Round Numerals for Results

- Give products, quotients, and powers to the same number of significant digits as appear in the least accurate approximation involved.
- Give sums and differences to the same number of decimal places as appear in the approximation with the least number of decimal places (the least precise measurement).

Example for 1-8

0.004006

4.006×10^{-3}

Example for 9-16

2.87×10^5

287,000

Example for 17-24

$$\frac{73.1\times(0.493)^2}{0.620\times(32.6)^2}=\frac{(7.31\times10^1)(4.93\times10^{-1})^2}{(6.20\times10^{-1})(3.26\times10^1)^2}$$

$$\frac{(7.31\times10^1)(4.93\times10^{-1})^2}{(6.20\times10^{-1})(3.26\times10^1)^2}=\frac{(7\times10^1)(5\times10^{-1})^2}{(6\times10^{-1})(3\times10^1)^2}$$

$$\frac{(7\times10^1)(5\times10^{-1})^2}{(6\times10^{-1})(3\times10^1)^2}=\frac{(7\times10^1)(25\times10^{-2})}{(6\times10^{-1})(9\times10^2)}$$

$$\frac{(7\times10^1)(25\times10^{-2})}{(6\times10^{-1})(9\times10^2)}=\frac{(7)(25)(10^{-1})}{(6)(9)(10^1)}=\frac{(7)(25)}{(6)(9)}\times10^{-2}$$

$$\frac{(7)(25)}{(6)(9)}\times10^{-2}=3.2\times10^{-2}=3\times10^{-2}=\ .03$$

Section 7-5

Decimal Numerals for Real Numbers

Objectives for Section 7-5

- to express fractions as decimals
- to find an intermediate rational and irrational number
- to express decimals as fractions

Decimal Numerals for Real Numbers

- rational numbers in their decimal form either repeat or terminate.
- density: a set S of real numbers is dense if between every two real numbers there is a member of S

Example for 1-8

$$\frac{13}{40}$$

$$40\overline{)13.000}\quad 0.325$$

0.325

Example for 9-12

$$\frac{7}{10} \text{ and } \frac{5}{8}$$

$$\frac{28}{40} \text{ and } \frac{25}{40}$$

a rational number would be $\frac{26}{40} = \frac{13}{20}$

$\frac{25}{40} = 0.625$ so an irrational number would be

0.62512345678910111213...

Example for 13-28

$$1.2\overline{567}$$

$$x = 1.2\overline{567}$$

$$10x = 12.\overline{567}$$

$$10{,}000x = 12567.\overline{567}$$

subtracting the bottom two equations yields

$$9{,}990x = 12{,}555$$

$$x = \frac{12{,}555}{9{,}990} = \frac{93}{74}$$

Section 7-6

Properties of Radicals

Objectives for Section 7-6

- to simplify radicals
- to express radicals in simplest form and give tenths and hundredths approximations
- to simplify variable radical expressions
- to prove statements about radicals

Roots

Index is even:

- base (radicand) > 0, then there are two real roots. The principal (positive) real root and the secondary (negative) real root.
  - When simplifying always use the principal root.
  - When solving you must give both roots.
- base (radicand) = 0, then there is one real root, 0.
- base (radicand) < 0, then there are no real roots.

Index is odd:

- base (radicand) > 0, then there is one real positive root.
- base (radicand) = 0, then there is one real root = 0.
- base (radicand) < 0, then there is one real negative root

Properties of Radicals

$$\left(\sqrt[n]{b}\right)^n = b$$

$$\sqrt[n]{b^n} = b, \text{ if } n \text{ is odd.}$$

$$\sqrt[n]{b^n} = |b|, \text{ if } n \text{ is even.}$$

Properties of Radicals

A radical is simplified if :

1. no radicand contains a factor other than 1 that has a power equal to the index
2. and every denominator has been rationalized so that no radicand is a fraction and no radical is in the denominator.

$$\sqrt[n]{ab} = \sqrt[n]{a} \bullet \sqrt[n]{b}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

$$\sqrt[nq]{b} = \sqrt[n]{\sqrt[q]{b}} \text{ or } \sqrt[q]{\sqrt[n]{b}}$$

$$\sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$$

Simplifying Radicals Review

Prime factor the radicand. Any factor which has an exponent equal to the index may be placed as a factor in front of the radical. All other factors must remain under the radical.

$$\sqrt[3]{1500} = \sqrt[3]{\left(5^3\right)\left(2^2\right)(3)} = 5\sqrt[3]{12}$$

Rationalizing Review

You must multiply the fraction by a form of the number one so that the radicand of the denominator has a power equal to its index.

$$\frac{\sqrt[4]{x^3}}{\sqrt[4]{y}}\left(\frac{\sqrt[4]{y^3}}{\sqrt[4]{y^3}}\right)=\frac{\sqrt[4]{x^3y^3}}{\sqrt[4]{y^4}}=\frac{\sqrt[4]{x^3y^3}}{|y|}$$

Example for 1-12

$$\sqrt[6]{(-64)^2}$$

$$\sqrt[6]{(-1)^2\left(4^3\right)^2}$$

$$\sqrt[6]{4^6}$$

$$|4| = 4$$

Example for 13-20

$$\sqrt{0.98}$$

$$\sqrt{\frac{98}{100}}$$

$$\frac{\sqrt{98}}{\sqrt{100}}$$

$$\frac{7\sqrt{2}}{10}$$

(0.7)(1.4) = 0.98

Example for 21-33

$$\sqrt{9x^4 + 9x^6 y^{-2}}$$

$$\sqrt{9x^4\left(1 + x^2 y^{-2}\right)}$$

$$\sqrt{9x^4\left(\frac{y^2 + x^2}{y^2}\right)}$$

$$\frac{3x^2\sqrt{x^2 + y^2}}{|y|}$$

Section 7-7

Operations with Radicals

Objectives for Section 7-7

- to simplify radical operations
- to rationalize denominators
- to factor over the set of polynomials with real coefficients

Radical Addition

- Radicals are considered similar and can be combined through addition and subtraction only if they have identical indices and identical radicands.
- If these two criteria are met, then the radicals can be combined and the number of similar radicals is expressed as a coefficient in front of the radical.

Conjugates

- To rationalize a fraction with a binomial radical in the denominator we will take advantage of the factoring pattern for (a + b)(a - b).
- Because this pattern has inside and outside products that cancel we will use factors that follow this pattern to eliminate the radicals.
- Terms like (a + b) & (a - b) are called conjugates.

Example for 1-10

$$\sqrt{18y^6} + y\sqrt{128y^4} - 5y^2\sqrt{162y^2}$$

$$3|y|^3\sqrt{2} + 8y^3\sqrt{2} - 45|y|^3\sqrt{2}$$

$$8y^3\sqrt{2} - 42|y|^3\sqrt{2}$$

Example for 11-20

$$\left(5\sqrt{2}-3\sqrt{7}\right)\left(5\sqrt{2}+3\sqrt{7}\right)$$

$$\left(5\sqrt{2}\right)\left(5\sqrt{2}\right)+\left(5\sqrt{2}\right)\left(3\sqrt{7}\right)+\left(5\sqrt{2}\right)\left(-3\sqrt{7}\right)+\left(3\sqrt{7}\right)\left(-3\sqrt{7}\right)$$

$$(25)(2)+15\sqrt{14}-15\sqrt{14}-(9)(7)$$

$$50-63=-13$$

Example for 21-24

$$\frac{4+\sqrt{5}}{6-2\sqrt{5}}$$

$$\left(\frac{4+\sqrt{5}}{6-2\sqrt{5}}\right)\left(\frac{6+2\sqrt{5}}{6+2\sqrt{5}}\right)$$

$$\frac{\left(4+\sqrt{5}\right)\left(6+2\sqrt{5}\right)}{36-20}$$

$$\frac{24+8\sqrt{5}+6\sqrt{5}+10}{16}$$

$$\frac{34+14\sqrt{5}}{16}=\frac{17+7\sqrt{5}}{8}$$

Example for 29-32

$$\frac{\sqrt[3]{5}}{\sqrt[3]{5}-\sqrt[3]{4}}$$

$$\left(\frac{\sqrt[3]{5}}{\sqrt[3]{5}-\sqrt[3]{4}}\right)\left(\frac{\sqrt[3]{25}+\sqrt[3]{20}+\sqrt[3]{16}}{\sqrt[3]{25}+\sqrt[3]{20}+\sqrt[3]{16}}\right)$$

$$\frac{\sqrt[3]{5}\left(\sqrt[3]{25}+\sqrt[3]{20}+\sqrt[3]{16}\right)}{5-4}$$

$$5+\sqrt[3]{100}+2\sqrt[3]{10}$$

Example for 33-40

$$c^3 + 6$$

Follow the pattern for $a^3 - b^3$ where c is a and $\sqrt[3]{6}$ is b.

$$\left(c + \sqrt[3]{6}\right)\left(c^2 - c\sqrt[3]{6} + \sqrt[3]{36}\right)$$

Section 7-8

Equations Involving Radicals

Objectives for Section 7-8

- to solve radical equations over the set of real numbers

Equations Involving Radicals

- For n a positive integer and a and b $\in \Re$
  - If a = b, then $a^n = b^n$.
  - If $a^n = b^n$ and n is odd, then a = b.
  - If $a^n = b^n$ and n is even, then a = ± b.

Solving with Radicals

I. Isolate the radical.

II. Raise both sides of the equation to the power of the index.

III. Solve the resulting equation by the appropriate method.

IV. If the equation has two radical terms, then perform steps I & II twice before solving the resulting equation.

V. Check answers in the original problem to identify any extraneous roots.

Example for 1-34 $\sqrt{2n-5}-\sqrt{3n+4}=2$

I. isolate the first radical $\sqrt{2n-5}=2+\sqrt{3n+4}$

II. square both sides and simplify

$$\left(\sqrt{2n-5}\right)^2=\left(2+\sqrt{3n+4}\right)^2$$

$$2n-5=4+4\sqrt{3n+4}+3n+4$$

$$2n-5=3n+8+4\sqrt{3n+4}$$

I. isolate the second radical $-n-13=4\sqrt{3n+4}$

II. square both sides and simplify

$$(-n-13)^2=\left(4\sqrt{3n+4}\right)^2$$

$$n^2+26n+169=16(3n+4)$$

$$n^2+26n+169=48n+64$$

$$n^2-22n+105=0$$

$$(n-7)(n-15)=0$$

$n = 7$ or 15 but neither answer works in the original problem so the final answer is ϕ

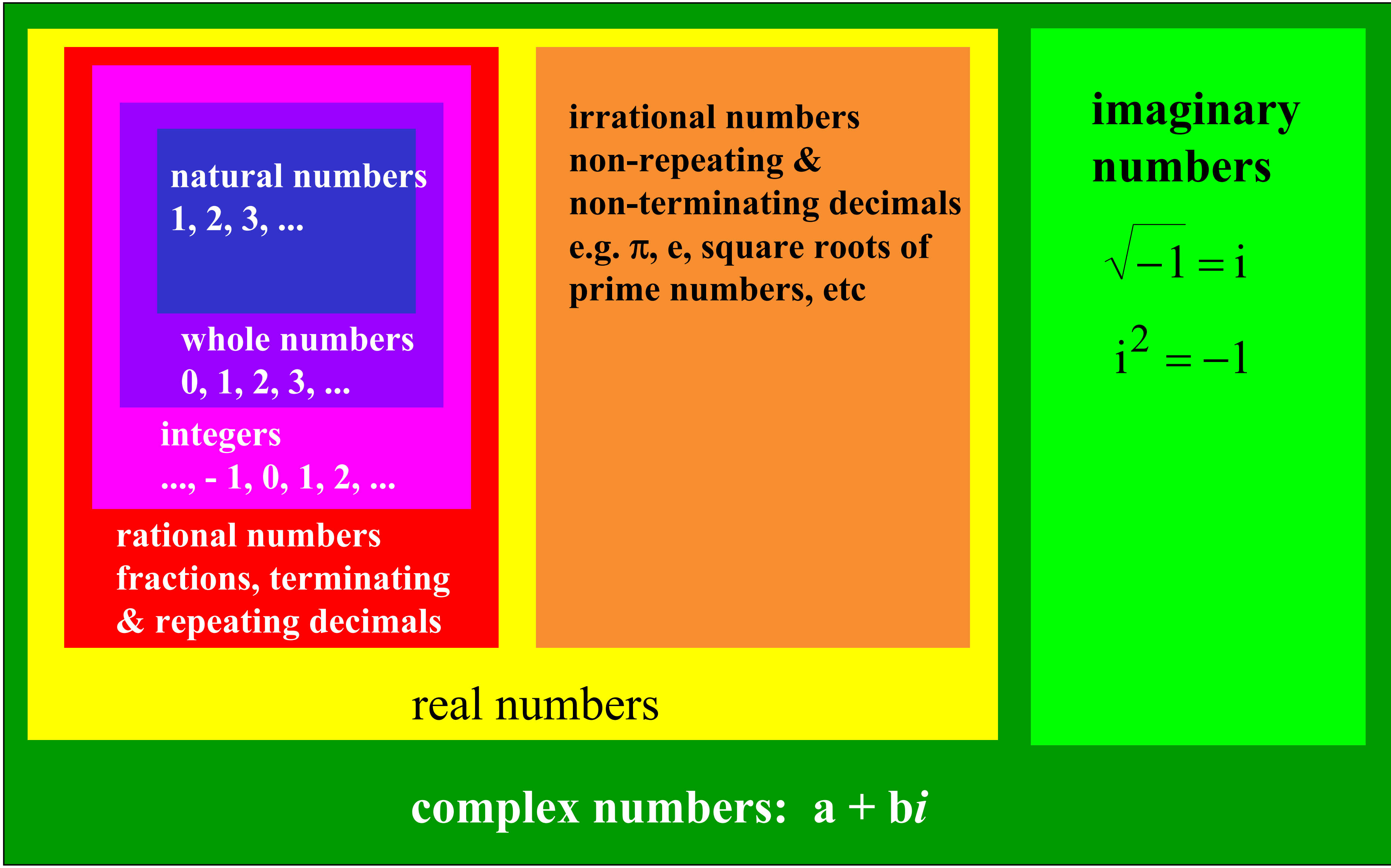
Section 7-9

The Number *i*

Objectives for Section 7-9

- to express imaginary numbers in simplest form

The Set of Complex Numbers



Properties of iota $(y\bar{o} - t\breve{a})$

- $\boldsymbol{i}^1 = \boldsymbol{i}$
- $\boldsymbol{i}^2 = -1$
- $\boldsymbol{i}^3 = -\boldsymbol{i}$
- $\boldsymbol{i}^4 = 1$
- the pattern repeats itself through these four values for all the powers of iota.
- Remember that an imaginary number is really a radical and cannot remain in the denominator of a fraction.

$$\frac{1}{\boldsymbol{i}} = -\boldsymbol{i}$$

$$\frac{1}{-\boldsymbol{i}} = \boldsymbol{i}$$

Example for 1-24

$$\frac{5}{\sqrt{-6}}$$

$$\frac{5}{\boldsymbol{i}\sqrt{6}}$$

$$\frac{5}{\boldsymbol{i}\sqrt{6}}\left(\frac{\boldsymbol{i}\sqrt{6}}{\boldsymbol{i}\sqrt{6}}\right)$$

$$\frac{5\boldsymbol{i}\sqrt{6}}{-6}$$

Example for 25-33

$$\sqrt{-x^5} + x\sqrt{-25x^2} - x^2\sqrt{-25x}$$

$$x^2 i\sqrt{x} + 5x^2 i - 5x^2 i\sqrt{x}$$

$$5x^2 i + \left(x^2 i\sqrt{x} - 5x^2 i\sqrt{x}\right)$$

$$5x^2 i - 4x^2 i\sqrt{x}$$

Section 7-10

Complex Numbers: Addition and Subtraction

Objectives for Section 7-10

- to perform operations with complex numbers
- to solve for the real and imaginary components of a complex number
- to prove statements about complex numbers

Addition and Subtraction of Complex Numbers

- When adding complex numbers add the real portions together and add the imaginary portions together to create a new complex number.
- If a, b, c and d are real numbers, then $a + b\boldsymbol{i} = c + d\boldsymbol{i}$ if and only if $a = c$ and $b = d$.
- If a, b, c and d are real numbers, then
  $(a + b\boldsymbol{i}) + (c + d\boldsymbol{i}) = (a + c) + (b + d)\boldsymbol{i}$
- For all real numbers a and b:

  $(a + b\boldsymbol{i}) + (a - b\boldsymbol{i}) = 2a$

  $(a + b\boldsymbol{i}) - (a - b\boldsymbol{i}) = 2b\boldsymbol{i}$
- complex conjugates: If $z = a + b\boldsymbol{i}$ then $\bar{z} = a - b\boldsymbol{i}$ and they are complex conjugates.

Example for 1-18

$a = 3 - 6\boldsymbol{i}$ and $d = -6\boldsymbol{i}$

$d - \overline{a}$

$-6\boldsymbol{i} - (3 + 6\boldsymbol{i})$

$-6\boldsymbol{i} - 3 - 6\boldsymbol{i}$

$\mathbf{-3 - 12}\boldsymbol{i}$

Example for 19-22

$$(x - y) + (x + y)\boldsymbol{i} = 5 + 9\boldsymbol{i}$$

$$x - y = 5$$
$$x + y = 9$$

$$2x = 14$$

$$\mathbf{x = 7, y = 2}$$

Section 7-11

Complex Numbers: Multiplication and Division

Objectives for Section 7-11

- to simplify products and quotients of complex numbers
- to factor over the set of complex polynomials

Multiplication and Division of Complex Numbers

- When multiplying complex numbers use the FOIL-ing process. If a, b, c and d are real numbers, then $(a + b\boldsymbol{i})(c + d\boldsymbol{i}) = (ac - bd) + (ad + bc)\boldsymbol{i}$.
- We know have a pattern for factoring sums of perfect squares $a^2 + b^2 = (a + b\boldsymbol{i})(a - b\boldsymbol{i})$
- Remember that an imaginary number is really a radical and cannot remain in the denominator of a fraction. Use the conjugate of a complex number to rationalize complex quotients.

Example for 1-7, 14-15

$$(-4 + \boldsymbol{i})(8 + 5\boldsymbol{i})$$

$$(-4)(8) + (-4)(5\boldsymbol{i}) + (\boldsymbol{i})(8) + (\boldsymbol{i})(5\boldsymbol{i})$$

$$-32 + (-20\boldsymbol{i}) + 8\boldsymbol{i} + 5\boldsymbol{i}^2$$

$$-32 - 12\boldsymbol{i} - 5$$

$$-37 - 12\boldsymbol{i}$$

Example for 9-12

$$\frac{5}{3+4i}$$

$$\left(\frac{5}{3+4i}\right)\left(\frac{3-4i}{3-4i}\right)$$

$$\frac{5(3-4i)}{25}$$

$$\frac{(3-4i)}{5}=\frac{3}{5}-\frac{4i}{5}$$

Example for 16-21

$$2w^2 + 98 = 0$$

$$2(w^2 + 49) = 0$$

$$2(w + 7\boldsymbol{i})(w - 7\boldsymbol{i}) = 0$$

$$w + 7\boldsymbol{i} = 0;\quad w - 7\boldsymbol{i} = 0$$

$$w = \pm 7\boldsymbol{i}$$