Section 9-1

Completing the Square and the Quadratic Formula

Objectives for 9-1

- to solve equations over the set of complex numbers by completing the square
- to solve equations over the set of complex numbers by using the quadratic formula
- to solve equations over the set of complex numbers by factoring

Quadratic Equations

- standard quadratic form is: $y = ax^2 + bx + c$
- When solving a quadratic equation remember that you are finding the x-intercepts of the graph so that $y = 0$.
- One way to solve an unfactorable quadratic equation is by completing the square.
  - move the constant to one side of the equation
  - divide through by (a) to make the quadratic coefficient 1
  - divide the linear coefficient (b) by 2, square this value and add it to both sides of the equation
  - factor the pattern polynomial that appears on one side of the equal sign and simplify the numerical values that appear on the other side of the equal sign
  - take the square root of both sides (remember to use both the principle and secondary roots since you are solving)
  - isolate the variable

Quadratic Formula

- If you were to complete the square on the formula: $0 = ax^2 + bx + c$ you would get

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example for 1-10, 21-44

$$5n^2 + 100 = 30n$$

$$5n^2 - 30n = -100$$

$$n^2 - 6n = -20$$

$$-6 \div 2 = -3, \ (-3)^2 = 9$$

$$n^2 - 6n + 9 = -20 + 9$$

$$(n-3)^2 = -11$$

$$\sqrt{(n-3)^2} = \sqrt{-11}$$

$$n - 3 = \pm i\sqrt{11}$$

$$n = 3 \pm i\sqrt{11}$$

Example for 11-20, 21-44

$$\frac{2m^2+16}{5}=2m$$

$$2m^2 + 16 = 10m$$

$$2m^2 - 10m + 16 = 0$$

$$a = 2,\ b = -10,\ c = 16$$

$$x=\frac{-(-10)\pm\sqrt{(-10)^2-4(2)(16)}}{2(2)}$$

$$x=\frac{10\pm\sqrt{100-128}}{4}$$

$$x=\frac{10\pm\sqrt{-28}}{4}$$

$$x=\frac{10\pm 2i\sqrt{7}}{4}$$

$$x=\frac{5\pm i\sqrt{7}}{2}\ \text{or}\ \frac{5}{2}\pm\frac{i\sqrt{7}}{2}$$

Section 9-2

The Discriminant

Objectives for 9-2

- to find the value of the discriminant and to use it to determine the nature of the roots of an equation
- to determine the value of a variable that will create an equation with a specified number of roots of a specified type

The Discriminant

- the discriminant $D = b^2 - 4ac$ will determine the nature of the roots of a quadratic equation in the form $y = ax^2 + bx + c$ only if a, b & c are all real numbers
- $D > 0$, then two unequal real roots
- $D = 0$, then one real double root
- $D < 0$, then two conjugate imaginary roots
- D is a perfect square and a, b & c are integers then roots are rational.

Example for 1-10

$$d^2 + \frac{7}{3}d = 2$$

$$3d^2 + 7d = 6$$

$$3d^2 + 7d - 6 = 0$$

$$D = (7)^2 - 4(3)(-6)$$

$$D = 49 + 72$$

$$\mathbf{D = 121}$$

real, unequal, rational

Example for 11-14

$k^2x^2 - 8x + 4 = 0$

it already is in standard quadratic form

$a = k^2$, $b = -8$ $c = 4$

$64 - 16k^2 = 0$

$k = \pm 2$

Example for 15 & 16

$k^2x^2 - 8x + 4 = 0$

it already is in standard quadratic form

$a = k^2$, $b = -8$ $c = 4$

$64 - 16k^2 > 0$

$(2 - k)(2 + k) > 0$

Example for 17 & 18

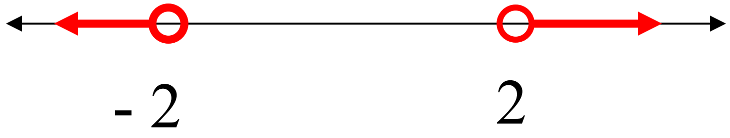
$$k^2x^2 - 8x + 4 = 0$$

it already is in standard quadratic form

$$a = k^2, b = -8 \quad c = 4$$

$$64 - 16k^2 < 0$$

$$(2 - k)(2 + k) < 0$$



Section 9-3

Roots and Coefficients of a Quadratic Equation

Objectives for 9-3

- to write a quadratic equation with a given solution set
- to find the value of a missing constant or coefficient in an equation so that it will have a specified root

Writing Standard Quadratic Equations

- If r_1 & r_2 are the roots of a quadratic equation, then that equation can be written as: $a[x^2 - (r_1 + r_2)x + r_1r_2] = y$

$$(r_1 + r_2) = -\frac{b}{a} \qquad\qquad r_1r_2 = \frac{c}{a}$$

- Roots, solutions, x-intercepts or zeros are synonyms.
- Identify which variables you have been given and substitute the values into the formula. You must know the values of (a), (r_1) and (r_2) to write the equation.
- Solve for the unknown value.
- Substitute in the values of (a), (r_1) and (r_2) and simplify.

Example for 1-12

$$\left\{\frac{4}{3}-\frac{2\boldsymbol{i}}{3},\frac{4}{3}+\frac{2\boldsymbol{i}}{3}\right\}$$

$$a[x^2 - (r_1 + r_2)x + r_1r_2] = y$$

$$a\left[x^2-\left(\frac{4}{3}-\frac{2\boldsymbol{i}}{3}+\frac{4}{3}+\frac{2\boldsymbol{i}}{3}\right)x+\left(\frac{4}{3}-\frac{2\boldsymbol{i}}{3}\right)\left(\frac{4}{3}+\frac{2\boldsymbol{i}}{3}\right)\right]=y$$

$$a\left[x^2-\left(\frac{8}{3}\right)x+\left(\frac{20}{9}\right)\right]=y$$

"a" is the LCD of the sum and product of the roots; therefore, a = 9.

$$9\left[x^2-\left(\frac{8}{3}\right)x+\left(\frac{20}{9}\right)\right]=y$$

$$9x^2-24x+20=y$$

Example for 13-18

One root of $x^2 - 4x + k = 0$ is $2+\sqrt{11}$

We know from our study of completing the square, the quadratic formula and the discriminant that the conjugate is the other root.

$$k = r_1 r_2 = \left(2+\sqrt{11}\right)\left(2-\sqrt{11}\right)$$

k = - 7

Section 9-4

The Graph of $(x - h)^2 = \pm 4p(y - k)$

Be able to graph all of the characteristics from the lesson and not just those asked for by the textbook.

Objectives for 9-4

- to graph quadratic functions and identify their essential characteristics
- to write a quadratic function for points described in a graph

Standard Parabolic Form

- all single variable quadratic equations describe a parabola
- parabolas can open in any direction but we will only be studying simple parabolas that open either up, down, left or right.
- Standard Parabolic Form is based on the definition of a parabola: the set of points equidistant from a given point (called the focus) and a given line (called the directrix). To see this derived look at the appendix for Chapter 7 for regular Algebra II.
  - opens up: $(x - h)^2 = 4p(y - k)$
  - opens down: $(x - h)^2 = - 4p(y - k)$
  - opens right: $(y - k)^2 = 4p(x - h)$
  - opens left: $(y - k)^2 = - 4p(x - h)$

Rearranging into Standard Parabolic Form

- Begin in standard quadratic form.
- If the equation equals zero, then remove the zero and replace with a (y).
- Complete the square.
  - $ax^2 + bx + c = y$
  - move the constant to the side with y.
  - divide by (a) so that the quadratic coefficient is 1
  - divide the linear coefficient by 2, square it, and add it to both sides of the equation.
  - factor the side with the variable into a binomial squared
- Factor out a GCF on the side with (y) so that the coefficient of (y) is 1.

Example of rearranging into Standard Parabolic Form

$$3x^2 - 2x + 1 = 0$$

$$3x^2 - 2x + 1 = y$$

$$3x^2 - 2x = y - 1$$

$$x^2 - \frac{2}{3}x = \frac{1}{3}y - \frac{1}{3}$$

$$\left[\left(-\frac{2}{3}\right)\left(\frac{1}{2}\right)\right]^2 = \left[-\frac{1}{3}\right]^2 = \frac{1}{9}$$

$$x^2 - \frac{2}{3}x + \frac{1}{9} = \frac{1}{3}y - \frac{1}{3} + \frac{1}{9}$$

$$\left(x - \frac{1}{3}\right)^2 = \frac{1}{3}y - \frac{2}{9}$$

$$\left(x - \frac{1}{3}\right)^2 = \frac{1}{3}\left(y - \frac{2}{3}\right)$$

standard parabolic form

Characteristics of a Parabola

1. coordinates of x -intercept(s)
2. coordinates of y-intercept(s)
3. coordinates of vertex
4. opens up, down, left or right
5. value of p
6. coordinates of focus
7. equation of the directrix
8. equation of the axis of symmetry
9. description of the Domain
10. description of the Range
11. maximum or minimum point

Locating and Graphing the Parts of Standard Parabolic Form

- Rearrange the equation into standard parabolic form by completing the square.
- Find and plot the x & y intercepts by substituting in zero.
- Determine if the parabola opens up, down, right or left.
- Identify the vertex, (h, k) and plot on a graph.
- Draw a rough sketch through the intercepts and vertex.
- Solve for (p).
- Find and plot the focus on the graph by moving (p) units from the vertex inside the parabola.
- Find and plot the equation of the directrix on a graph by moving (p) units from the vertex outside the parabola to either a horizontal (opens up/down) or vertical (opens right/left) line.
- Find and plot the equation of the axis of symmetry on the graph by connecting the vertex and the focus.
- From the graph describe the Domain and Range.
- Identify, if any, the maximum or minimum value of the graph.

Finding the Intercepts

- For a parabola the intercept(s) are where the graph crosses the x and y-axis.
- Remember any point on the x-axis has a y-value of 0 and any point on the y-axis has an x value of 0.

Example of locating x-intercept(s)

$$\left(x-\frac{1}{3}\right)^2=\frac{1}{3}\left(y-\frac{2}{3}\right)$$

$$\left(x-\frac{1}{3}\right)^2=\frac{1}{3}\left(0-\frac{2}{3}\right)$$

$$\left(x-\frac{1}{3}\right)^2=\frac{1}{3}\left(-\frac{2}{3}\right)$$

$$\left(x-\frac{1}{3}\right)^2=-\frac{2}{9}$$

$$\sqrt{\left(x-\frac{1}{3}\right)^2}=\sqrt{-\frac{2}{9}}$$

$$x-\frac{1}{3}=\pm\frac{i\sqrt{2}}{3}$$

$$x=\frac{1}{3}\pm\frac{i\sqrt{2}}{3}$$

X-Intercepts are the points:

$$\left(\frac{1}{3}+\frac{i\sqrt{2}}{3},0\right)\&\left(\frac{1}{3}-\frac{i\sqrt{2}}{3},0\right)$$

The parabola does not cross the x-axis because the intercepts are imaginary.

Example of locating y-intercept(s)

$$\left(x-\frac{1}{3}\right)^2=\frac{1}{3}\left(y-\frac{2}{3}\right)$$

$$\left(0-\frac{1}{3}\right)^2=\frac{1}{3}\left(y-\frac{2}{3}\right)$$

$$\left(-\frac{1}{3}\right)^2=\frac{1}{3}\left(y-\frac{2}{3}\right)$$

$$\frac{1}{9}=\frac{1}{3}\left(y-\frac{2}{3}\right)$$

$$\frac{1}{3}=y-\frac{2}{3}$$

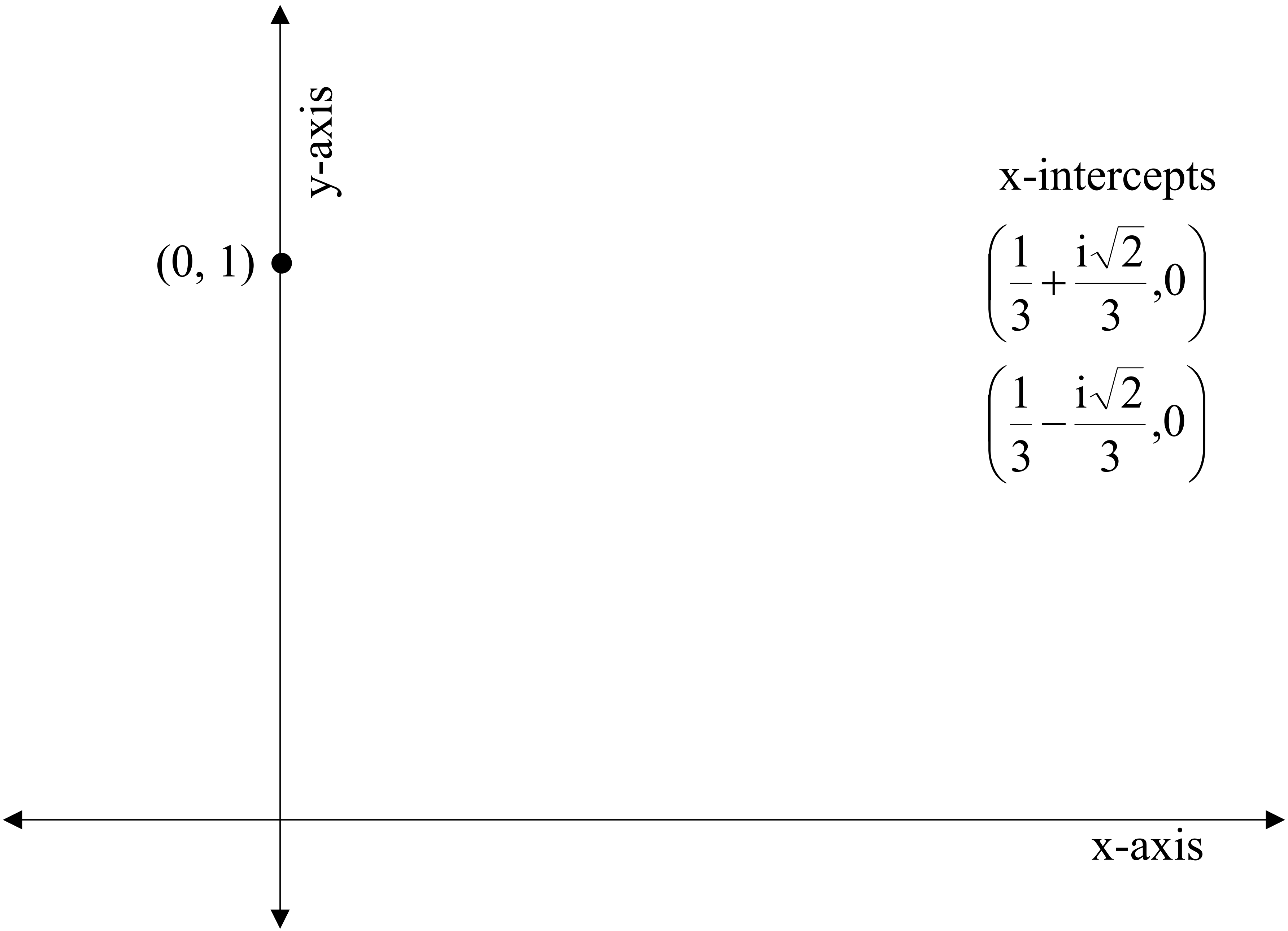
$$y=1$$

y-intercept is the point

$$(0,1)$$

The parabola crosses the y-axis once.

Example of graphing intercepts



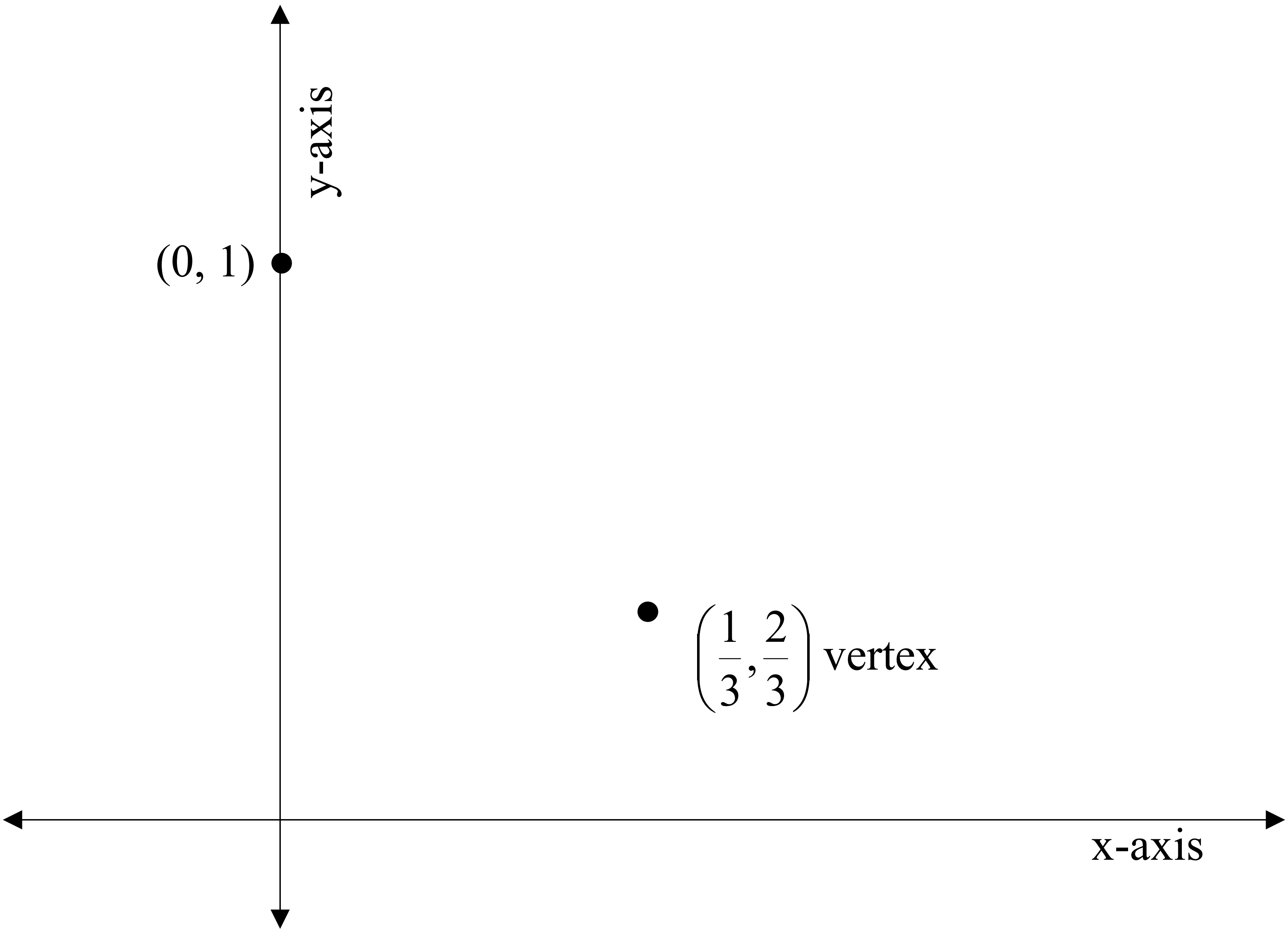
Example of identifying the vertex

$$\left(x-\frac{1}{3}\right)^2=\frac{1}{3}\left(y-\frac{2}{3}\right)$$

$$(x-h)^2=4p(y-k)$$

$$\text{vertex}\left(\frac{1}{3},\frac{2}{3}\right)$$

Example of graphing the vertex



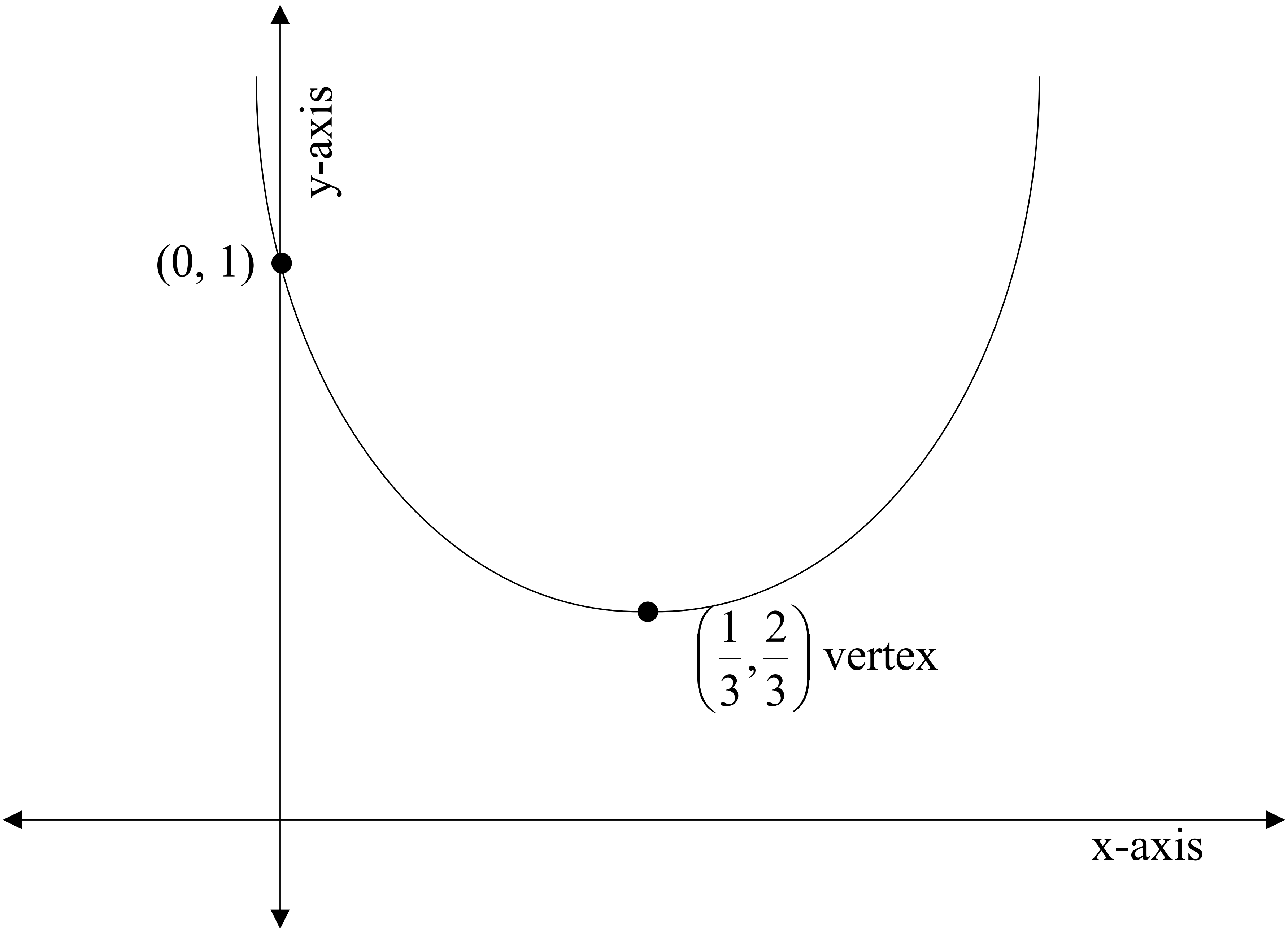
Example of determining the direction of the parabola

- Once it is in standard parabolic form look at the number outside the parentheses and the variable inside the parentheses on the linear side of the equation:

$$\left(x-\frac{1}{3}\right)^2=\frac{1}{3}\left(y-\frac{2}{3}\right)$$

- The number outside the parentheses is positive and the variable inside is y. Because positive y values are "up" on a graph, we know that the parabola opens up.

Example of graphing the direction of a parabola



Example of solving for the value of p

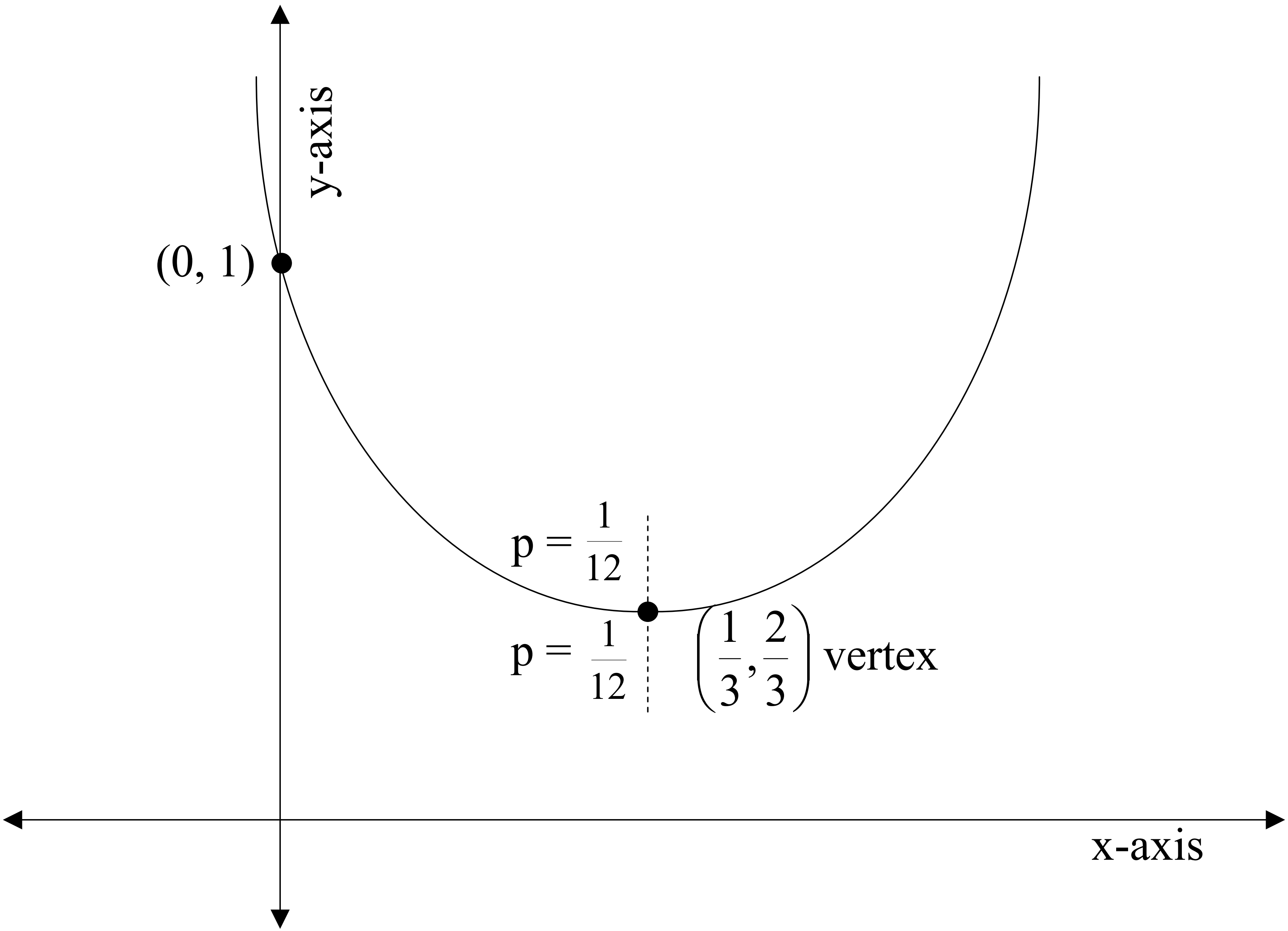
$$(x-h)^2 = 4p(y-k)$$

$$\left(x-\frac{1}{3}\right)^2 = \frac{1}{3}\left(y-\frac{2}{3}\right)$$

$$4p = \frac{1}{3}$$

$$p = \frac{1}{12}$$

Example of graphing p



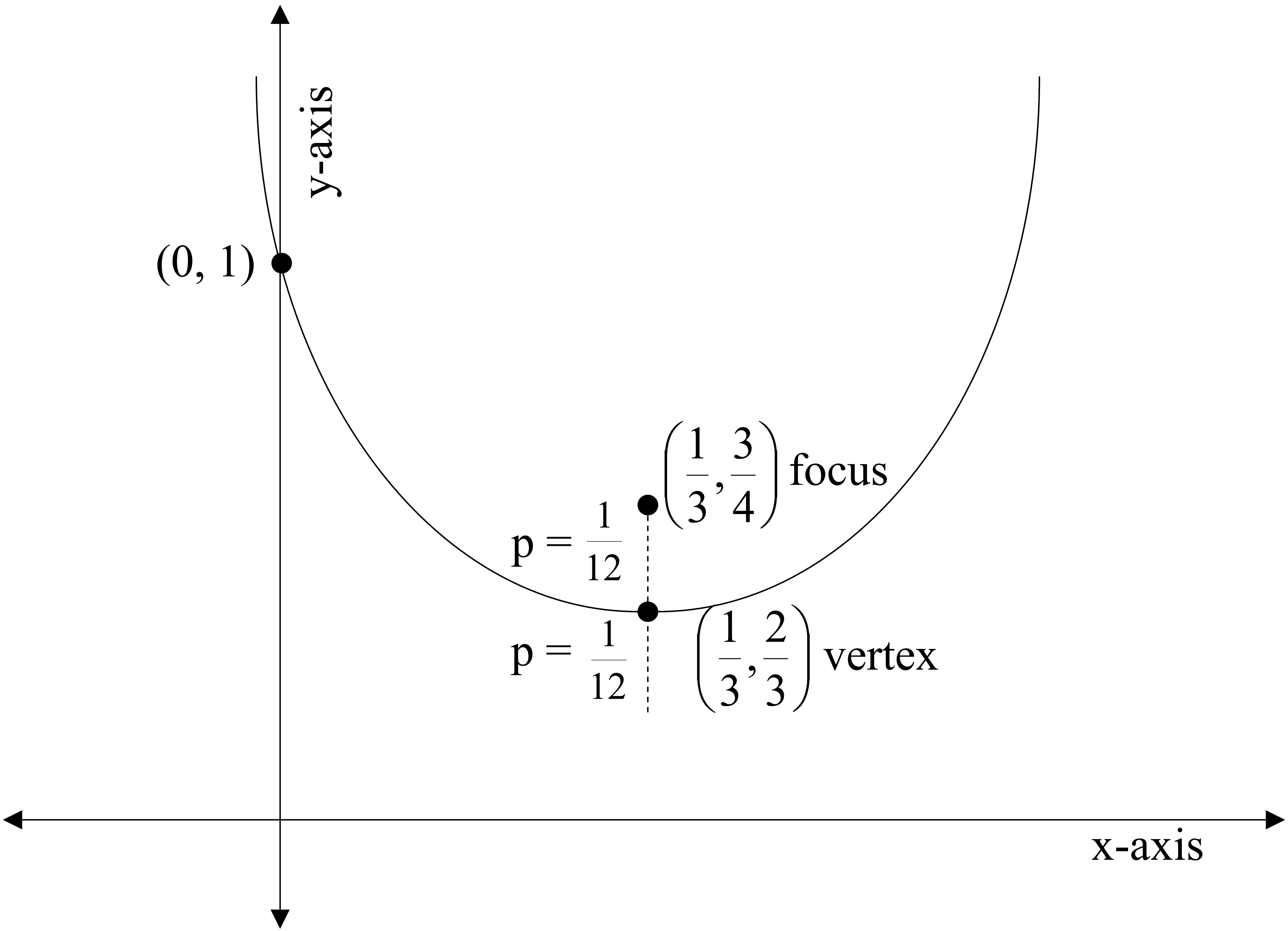
Example of locating the focus

$$\text{vertex}\left(\frac{1}{3},\frac{2}{3}\right) \quad p=\frac{1}{12}$$

$$\text{focus}\left(\frac{1}{3},\frac{2}{3}+\frac{1}{12}\right)$$

$$\text{focus}\left(\frac{1}{3},\frac{3}{4}\right)$$

Example of graphing the focus



Example of locating the directrix & axis of symmetry

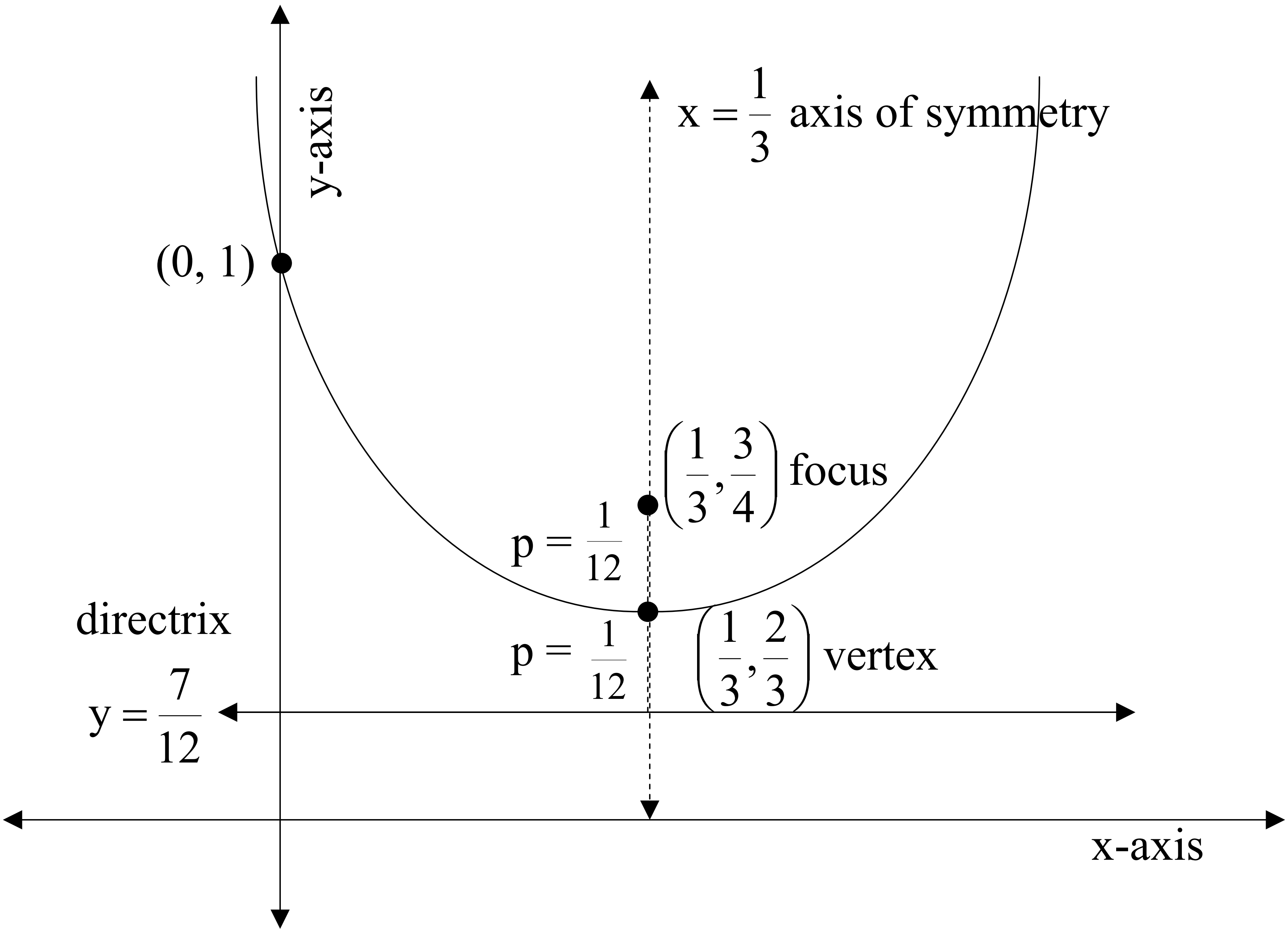
$$\text{vertex}\left(\frac{1}{3},\frac{2}{3}\right) \quad p=\frac{1}{12}$$

directrix : horizontal line with the equation $y=\frac{2}{3}-\frac{1}{12}$

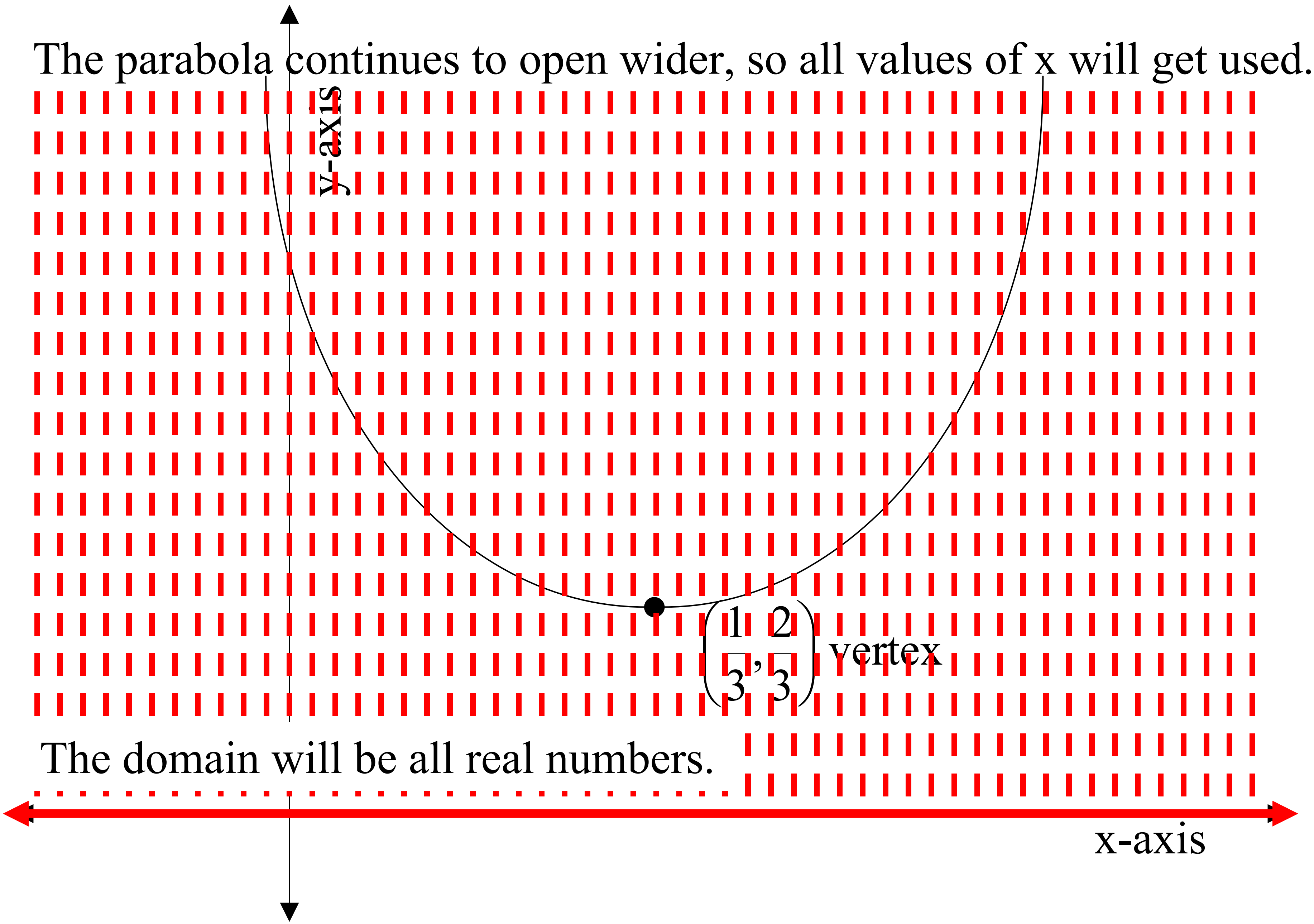
$$y=\frac{7}{12}$$

axis : vertical line through the vertex with the equation $x=\frac{1}{3}$

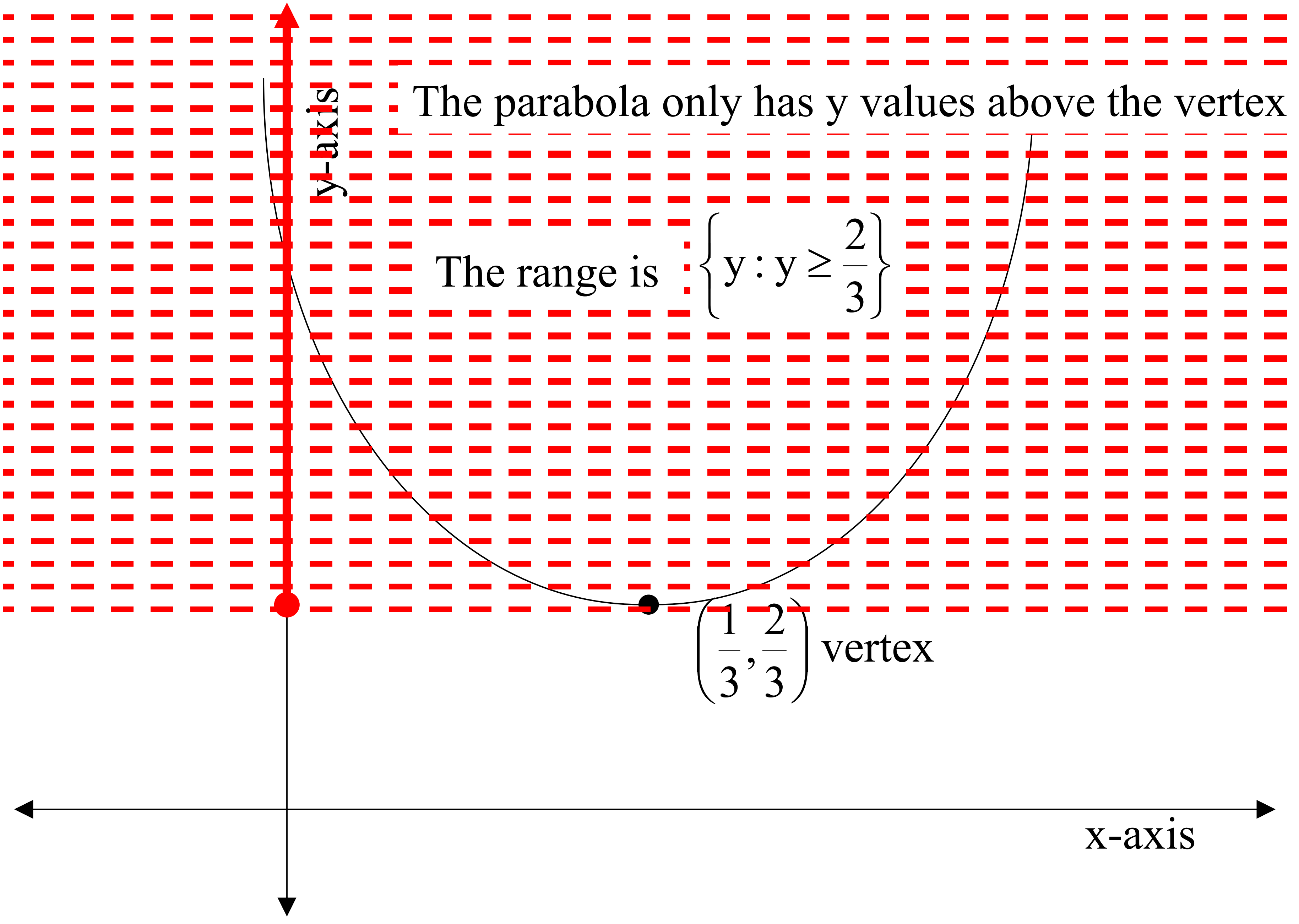
Example of graphing the directrix & the axis of symmetry



Example of describing the domain



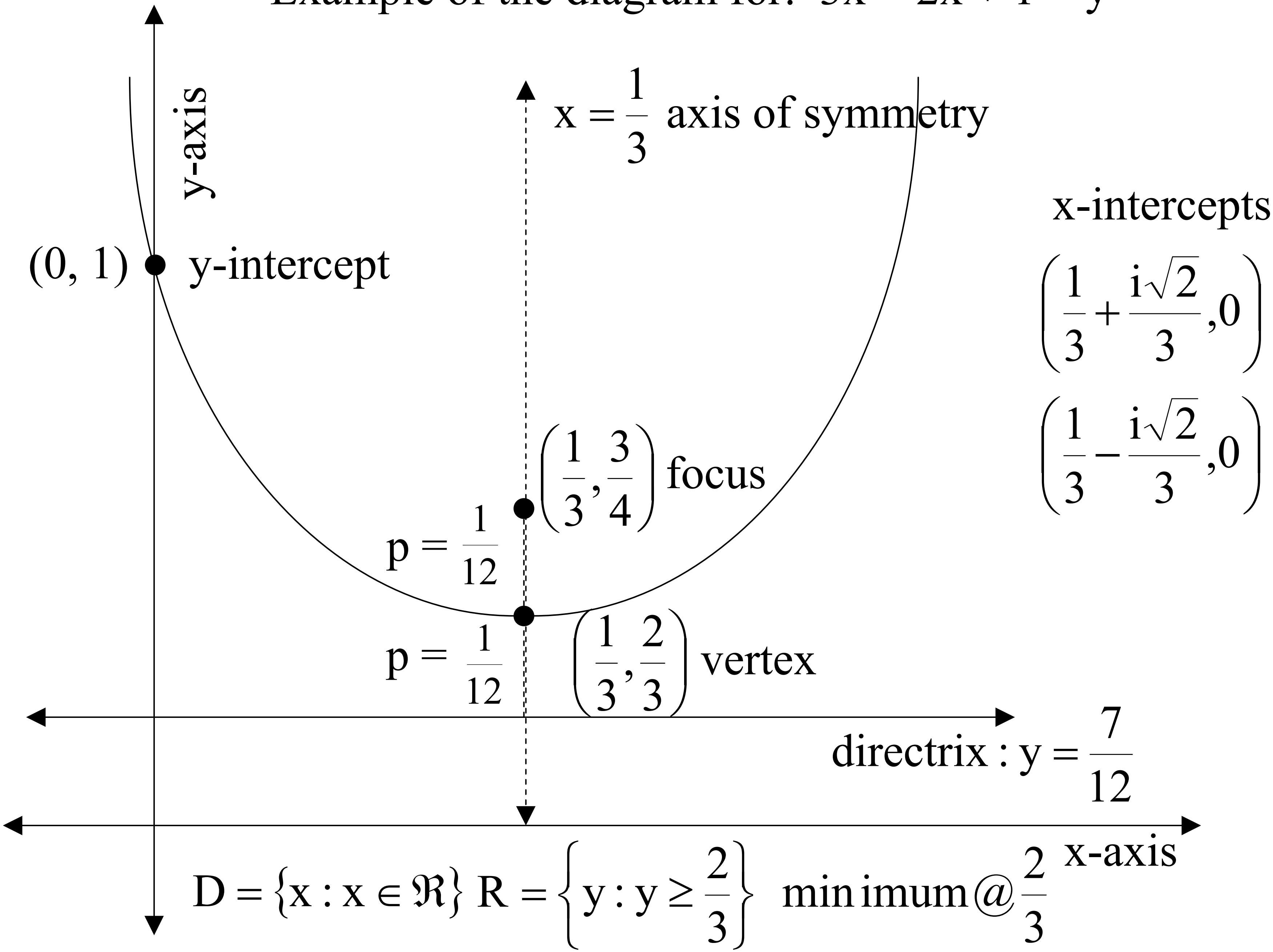
Example of describing the range



Example of finding the maximum and minimum values

- The value of an equation is an element of the range.
- A maximum is the largest y value: a minimum is the smallest y value.
- A graph has either a maximum, a minimum or neither.
- This graph has a minimum y value of $\frac{2}{3}$

Example of the diagram for: $3x^2 - 2x + 1 = y$

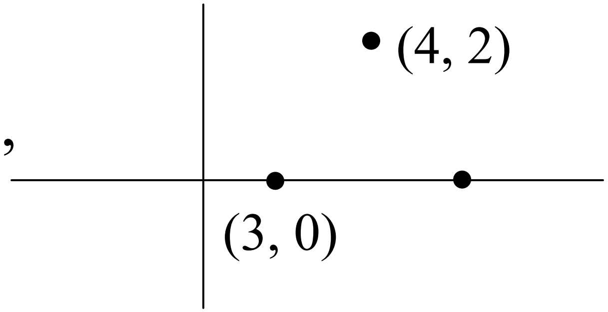


Writing Standard Parabolic Equations

- plot the information given on a graph to determine the direction of the parabola
- based on the direction, select the appropriate formula
  - opens up: $(x - h)^2 = 4p(y - k)$
  - opens down: $(x - h)^2 = - 4p(y - k)$
  - opens right: $(y - k)^2 = 4p(x - h)$
  - opens left: $(y - k)^2 = - 4p(x - h)$
- substitute the values from your graph into the formula for the corresponding variables
- solve for the missing variable
- the equation is complete when you have the values of (h), (k), and (p)

Example of writing a Standard Parabolic Equation

- Find the standard parabolic equation of a parabola with a vertex (4, 2) and one x intercept 3.
- From a rough graph you can tell that the parabola is opening down; therefore, you would use the equation
  $(x - h)^2 = - 4p(y - k)$



- h = 4, k = 2, x = 3 & y = 0; therefore, you must solve for p
  $(3 - 4)^2 = - 4p(0 - 2)$
  $(- 1)^2 = (- 4)(- 2)p$
  $1 = 8p$
  $p = \frac{1}{8}$
- the equation will be: $(x - 4)^2 = -\frac{1}{2}(y - 2)$

Section 9-5

The Graph of a Quadratic Function

Be able to graph all of the characteristics from the lesson and not just those asked for by the textbook.

Objectives for 9-5

- to graph quadratic functions and identify their essential characteristics
- to write a quadratic function for points described in a graph

Finding Maximum & Minimum Points

- For any standard form quadratic (not parabolic) equation in the form $y = ax^2 + bx + c$ or $f(x) = ax^2 + bx + c$:
  - if $a < 0$, then the parabola opens down and the vertex is the maximum point.
  - if $a > 0$, then the parabola opens up and the vertex is the minimum point.
- The coordinates of the vertex can be written as the point:

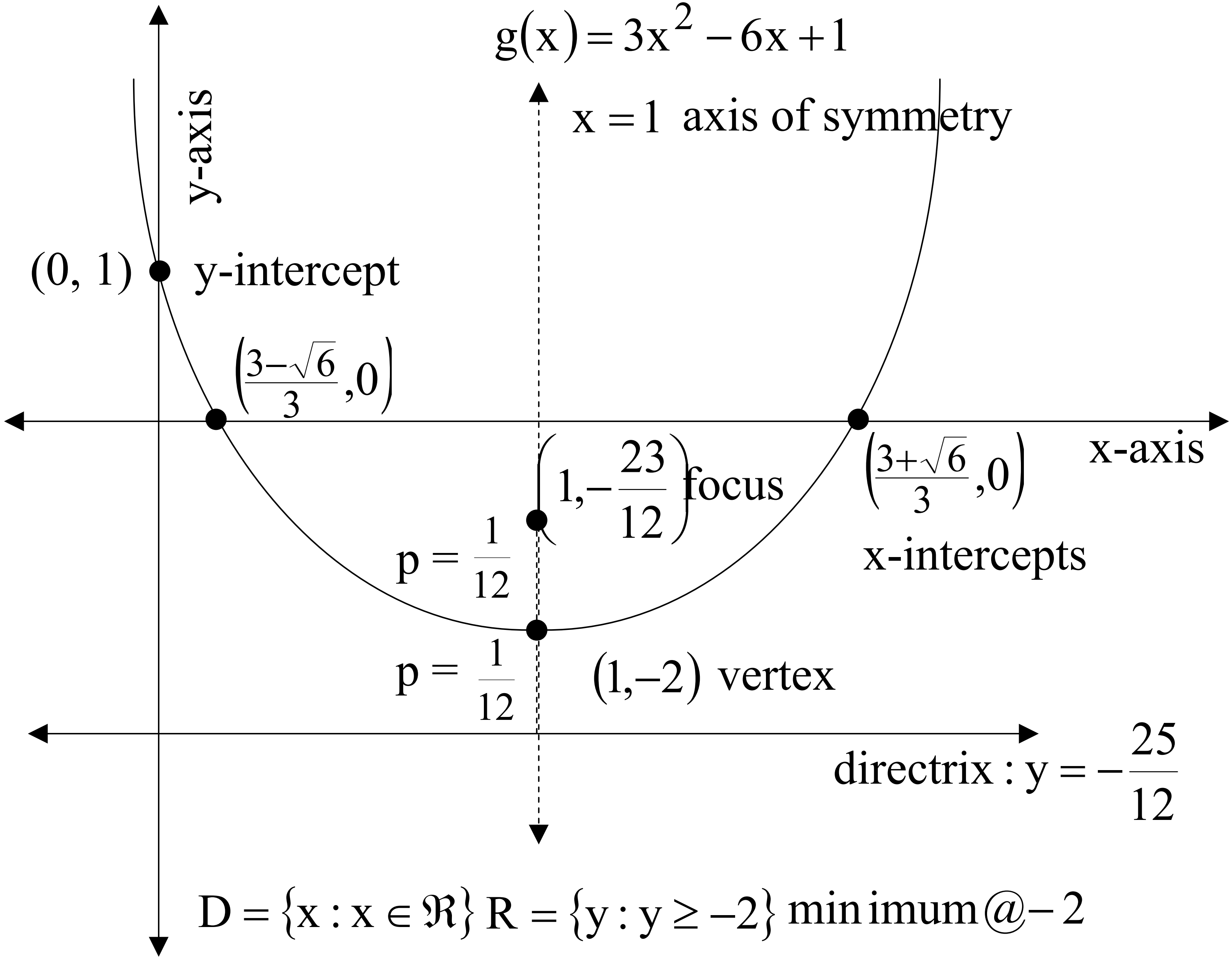
$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) = \left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$$

- The distance "p" from the vertex to the focus or the distance "p" from the vertex to the directrix is:

$$\left|\frac{1}{4a}\right|$$

Example for 1-12

$$g(x) = 3x^2 - 6x + 1$$



$D = \{x : x \in \Re\}$ $R = \{y : y \geq -2\}$ minimum @ -2

Example for 13-16

(- 3, 2) (- 1, 6) (1, 2)

$$2 = 9a - 3b + c$$
$$6 = a - b + c$$
$$2 = a + b + c$$

$$4 = -2b$$
$$-2 = b$$

$$0 = 8a - 4b$$
$$0 = 8a + 8$$
$$-1 = a$$

$$2 = -1 - 2 + c$$
$$5 = c$$

$$\mathbf{y = -x^2 - 2x + 5}$$

Example for 17-20

$$rx^2 + 2rx = y$$

$$r(x^2 + 2x) = y$$

$$r(x^2 + 2x + 1) = y + r$$

$$r(x + 1)^2 = y + r$$

$$(x+1)^2 = \frac{1}{r}(y+r)$$

see next slide

x-intercepts: (0, 0) (- 2, 0)

y-intercept: (0, 0)

vertex (- 1, - r)

direction: If r > 0, then up; if r < 0, then down.

value of p: $\left|\frac{1}{4r}\right|$

coordinates of focus: $r > 0, \left(-1, -r + \left|\frac{1}{4r}\right|\right); r < 0, \left(-1, -r - \left|\frac{1}{4r}\right|\right)$

equation of the directrix: $r > 0, y = -r - \left|\frac{1}{4r}\right|; r < 0, y = -r + \left|\frac{1}{4r}\right|$

equation of the axis: x = - 1

Domain: {x: x $\in \Re$}

Range: {y: if r > 0, then y ≥ - r; if r < 0, then y ≤ - r}

If r > 0, then minimum @ - r; if r < 0, then maximum @ - r.

Section 9-6

Quadratic Inequalities

Objectives for 9-6

- to graph the solution set to a quadratic inequality

Quadratic Inequalities

- To find the solution to a quadratic inequality you must first find the roots of the quadratic equation.
- Next use "a" to determine whether the parabola opens up or down.
- Combined these two pieces of information can tell you where the graph is above the x-axis (positive values) and below the x-axis (negative values).

Example for 1-9

$$2x^2 + 7 \le 9x$$

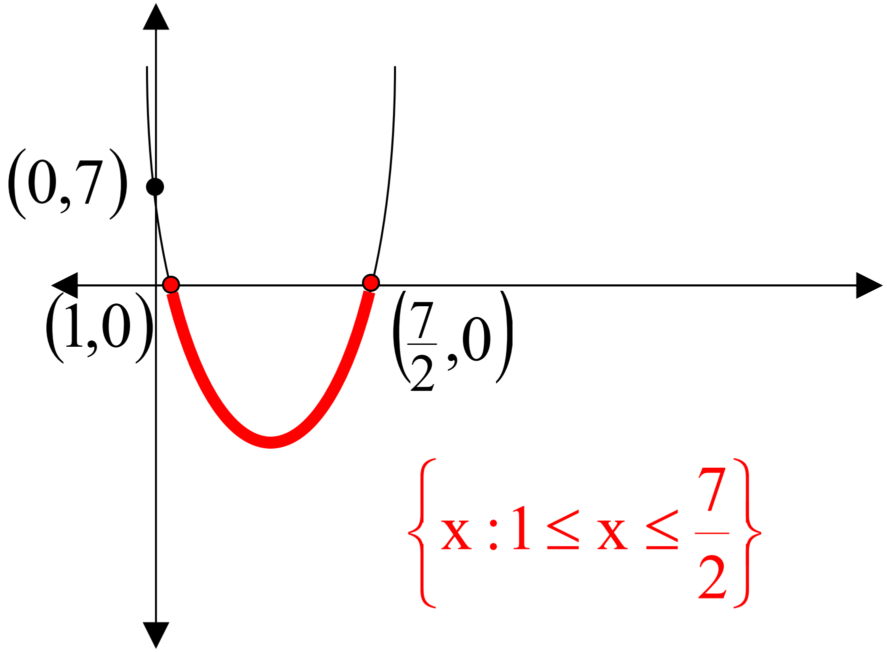
$$2x^2 - 9x + 7 \le 0$$

$$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(2)(7)}}{2(2)}$$

$$x = \frac{9 \pm \sqrt{81 - 56}}{4}$$

$$x = \frac{9 \pm \sqrt{25}}{4}$$

$$x = \frac{9 \pm 5}{4} = \frac{7}{2} \text{ or } 1$$



$$\left\{x : 1 \le x \le \frac{7}{2}\right\}$$

a is positive therefore the graph opens up and its graph is below the x-axis (negative values) between the intercepts

Section 9-7

Values of Polynomial Functions

Objectives for 9-7

- to use synthetic substitution to find the value of a function over the set of complex numbers
- to use the rational root theorem and synthetic substitution to locate solutions to a polynomial equation

Values of Polynomial Functions

- You know already that if you want to find the value (y) of a polynomial function for a given value of (x) you can substitute x directly into the function and simplify.
- Another method to find the value of a function is through the process of synthetic substitution.
- To perform synthetic substitution the polynomial must be written in descending order by degree and any missing powers of the variable must be filled in with zeroes.
- The process of synthetic substitution is best explained with the following example.

Example for 1-20

$$P(x) = 2x^4 - 7x^3 + 7x + 6 \quad P(3) = ?$$

3	2	-7	0	7	6
		6	-3	-9	-6
	2	**-1**	**-3**	**-2**	**0**

quotient: $2x^3 - x^2 - 3x - 2$

remainder: the value of the polynomial at $x = 3$

P(3) = 0

Synthetic Division

- In the previous example the polynomial P(x) was being divided by the factor (x – 3) just as we did when we performed polynomial long division.
- The value P(3) = 0 would have been the remainder from the long division process indicating that (x – 3) is a factor of P(x).
- The other factor was the quotient polynomial $Q(x) = 2x^3 - x^2 - 3x - 2$.
- When synthetic division is combined with the rational root theorem it provides a relatively fast way to locate the rational roots of a polynomial equation.

Example for 21-24

3	2	- 7	5	m
		6	- 3	6
	2	- 1	2	- 1

$$m + 6 = -1$$

$$\mathbf{m = -7}$$

Example for 25-28

$$x^3 - 8x^2 + 5x + 14 = 0$$

$$\frac{\pm1 \pm2 \pm7 \pm14}{\pm1} = \{\pm1 \pm2 \pm7 \pm14\}$$

-1	1	-8	5	14
		-1	9	-14
	1	-9	14	0

7	1	-8	5	14
		7	-7	-14
	1	-1	-2	0

2	1	-8	5	14
		2	-12	-14
	1	-6	-7	0

{x: - 1, 2, 7}

Section 9-8

Remainder and Factor Theorems

Objectives for 9-8

- to use synthetic substitution to change the form of a polynomial equation
- to use synthetic substitution to solve polynomial equations of the third degree

Remainder and Factor Theorems

- Remainder Theorem: For every polynomial $P(x)$ of positive degree n over the set of complex numbers, and for every complex number r, there exists a polynomial $Q(x)$ of degree $n - 1$, such that $P(x) = (x - r)Q(x) + P(r)$
- If $P(x) \div (x - r) = 0$, then $Q(x)$ is called the depressed equation of $P(x) = 0$. The depressed equation is used to solve for the remaining roots of $P(x)$.
- Factor Theorem: Over the set of complex numbers, $x - r$ is a factor of a polynomial $P(x)$ if and only if r is a root of $P(x) = 0$.

Example for 1-4

$$P(x) = 3x^3 – 8x^2 + 5x + 6,\ r = 2$$

2	3	- 8	5	6
		6	- 4	2
	3	- 2	1	8

$$\mathbf{3x^3 – 8x^2 + 5x + 6 = (x – 2)(3x^2 – 2x + 1) + 8}$$

Example for 5-8

$$P(x) = x^3 - 13x + 18,\ x + 4$$

-4	1	0	-13	18
		-4	16	-12
	1	-4	3	6

$$\frac{x^3 - 13x + 18}{x + 4} = x^2 - 4x + 3 + \frac{6}{x + 4}$$

Example for 9-14

$x - 2\boldsymbol{i}$; $P(x) = 2x^3 - 5x^2 + 8x + 20$

$2\boldsymbol{i}$				
	2	-5	8	20
		$4\boldsymbol{i}$	$-8-10\boldsymbol{i}$	20
	2	$-5+4\boldsymbol{i}$	$-10\boldsymbol{i}$	40

no, 40

Example for 15-22

$$x^3 + 3x^2 - 2x - 6;\ r_1 = -3$$

-3	1	3	-2	-6
		-3	0	6
	1	0	-2	0

$$x = \frac{-0 \pm \sqrt{0^2 - 4(1)(-2)}}{2(1)} = \pm\sqrt{2}$$

Example for 23-28

$$P(x) = x^3 + 4x^2 + x - 6$$

$$\frac{\pm 1 \pm 2 \pm 3 \pm 6}{\pm 1} = \{\pm 1 \pm 2 \pm 3 \pm 6\}$$

1	1	4	1	- 6
		1	5	6
	1	5	6	0

$$x = \frac{-5 \pm \sqrt{5^2 - 4(1)(6)}}{2} = -2, -3$$

{x: 1, - 2, - 3}

Example for 29-31

Find m such that – 2 will be a zero of $x^3 + 3x^2 - 7x + m$.

- 2	1	3	- 7	m
		- 2	- 2	18
	1	1	- 9	0

$$m + 18 = 0$$

m = - 18

Section 9-9

The Fundamental Theorem of Algebra

Objectives for 9-9

- to write a polynomial equation with given roots
- to find all of the roots of a polynomial equation when given one root and to factor the polynomial completely

Fundamental Theorem of Algebra

- Every polynomial equation with complex coefficients and positive degree n has exactly n complex roots.
- If a polynomial equation with real coefficients has a + b***i*** as a root (a and b real, b ≠ 0), then a – b***i*** is also a root.

Example for 1-6

$$6, 1 + \boldsymbol{i}\sqrt{5}, 1 - \boldsymbol{i}\sqrt{5}$$

$$(x - 6)(x - (1 + \boldsymbol{i}\sqrt{5}))(x - (1 - \boldsymbol{i}\sqrt{5}))$$

$$(x - 6)(x^2 - 2x + 6)$$

$$x^3 - 8x^2 + 18x - 36 = 0$$

Example for 7-14

$$x^4 + 11x^2 + 18; \ -3\boldsymbol{i}$$

If $-3\boldsymbol{i}$ is a root then $3\boldsymbol{i}$ is also a root and $x^2 + 9$ is a factor.

$$\begin{array}{r} x^2 + 2 \\ x^2 + 0x + 9 \overline{\big) x^4 + 0x^3 + 11x^2 + 0x + 18} \\ \underline{x^4 + 0x^3 + 9x^2} \qquad\qquad \\ 2x^2 + 0x + 18 \\ \underline{2x^2 + 0x + 18} \\ 0 \end{array}$$

$$\left\{x : 3\boldsymbol{i}, -3\boldsymbol{i}, \boldsymbol{i}\sqrt{2}, -\boldsymbol{i}\sqrt{2}\right\}$$

Section 9-10

Locating Possible Real Roots

Objectives for 9-10

- to use Descartes rule of signs to discover the possible number and types of roots for a polynomial equation
- to find the least positive integral upper bound and the greatest negative integral lower bound of a polynomial equation

Descartes' Rule of Signs

- If P(x) is a polynomial with real coefficients, then
  - the number of positive roots of P(x) = 0 is either equal to the number of variations of sign of P(x) or is less than this number by a positive even integer;
  - the number of negative roots of P(x) is either equal to the number of variations of sign for P(- x) or is less than this by a positive even integer.

Upper and Lower Bounds for Roots

- Let P(x) be a polynomial with real coefficients and with positive leading coefficient.
  - If $M \geq 0$ and the coefficients of the quotient and remainder obtained on dividing P(x) by $x - M$ are all greater than or equal to 0, then P(x) has no roots greater than M.
  - If $L \leq 0$ and the coefficients and remainder on dividing P(x) by $x - L$ are alternately greater than or equal to 0 and less than or equal to 0, then P(x) has no roots less than L.
- M and L are called the upper and lower bounds respectively, for the roots of the given polynomial.

Example for 1-6

- $x^4 + 2x^3 + x^2 + 1 = 0$
- If x is positive then the terms are: (+) (+) (+) (+). There are no sign changes so there are no positive real roots.
- If x is negative then the terms are (+) (-) (+) (+). There are two sign changes so there are either 2 negative real roots or no negative real roots.
- **The polynomial is degree 4 so there are 4 roots:**
  - **no positive real roots**
  - **two negative real root**
  - **two imaginary roots**

  OR
  - **no positive real roots**
  - **no negative real roots**
  - **four imaginary roots**

Example for 7-14

1	1	-2	-1	-3
		1	-1	-2
	1	-1	-2	-5

-1	1	-2	-1	-3
		-1	3	-2
	1	**-3**	**2**	**-5**

2	1	-2	-1	-3
		2	0	-2
	1	0	-1	-5

3	1	-2	-1	-3
		3	3	6
	1	**1**	**2**	**3**

Since the coefficients of the depressed equation and the remainder are all positive values, 3 is the upper bound of the roots. Also, since the coefficients of the depressed equation and the remainder alternate signs, - 1 is the lower bound of the roots.

Section 9-11

Estimates of Real Roots

Objectives for 9-11

- to estimate the roots of a polynomial equation to the nearest half unit
- to identify all of the half unit intervals containing a root of a polynomial equation
- to use linear interpolation to estimate roots to the nearest hundredth

Estimates of Real Roots

- Intermediate Value Property: If P is a polynomial function with real coefficients, and if m is any number between P(a) and P(b), then there is at least one number c between a and b for which P(c) = m.
- linear interpolation: the process of evaluating P(x) by using a line segment.

Example for Estimating Roots

$$P(x) = x^3 - 3x + 1$$

{x: ______, ______, ______ }

$$\frac{\pm 1}{\pm 1} = \{\pm 1\}$$

+	–	*i*
2	1	0
0	1	2

- 1	1	0	- 3	1
		- 1	1	2
	1	- 1	- 2	3

- 2	1	0	- 3	1
		- 2	4	- 2
	1	- 2	1	- 1

There is a root between – 1 and – 2, and – 2 is the lower bound.

1	1	0	- 3	1
		1	1	- 2
	1	1	- 2	- 1

2	1	0	- 3	1
		2	4	2
	1	2	1	3

There are roots between 0 and 1 and between 1 and 2. Also 2 is the upper bound.

$$
\begin{array}{r|rrrr}
-\frac{3}{2} & 1 & 0 & -3 & 1 \\
 & & -\frac{3}{2} & \frac{9}{4} & \frac{9}{8} \\
\hline
 & 1 & -\frac{3}{2} & -\frac{3}{4} & \frac{17}{8} \approx 2.125
\end{array}
$$

$$
\begin{array}{r|rrrr}
\frac{1}{2} & 1 & 0 & -3 & 1 \\
 & & \frac{1}{2} & \frac{1}{4} & -\frac{11}{8} \\
\hline
 & 1 & \frac{1}{2} & -\frac{11}{4} & -\frac{3}{8} \approx -0.375
\end{array}
$$

$$
\begin{array}{r|rrrr}
\frac{3}{2} & 1 & 0 & -3 & 1 \\
 & & \frac{3}{2} & \frac{9}{4} & -\frac{9}{8} \\
\hline
 & 1 & \frac{3}{2} & -\frac{3}{4} & -\frac{1}{8} \approx -0.125
\end{array}
$$

x	P(x)
- 2	- 1
- 1.5	2.125
- 1	3
0	1
0.5	- 0.375
1	- 1
1.5	- 0.125
2	3

x	P(x)
- 2	- 1
-1.84	0
- 1.5	2.125
- 1	3
0	1
0.36	0
0.5	- 0.375
1	- 1
1.5	- 0.125
1.52	0
2	3

Interval from x = - 2 to x = - 1.5: interval width 0.5, Δ x; P(x) changes by 1 (to the root), 3.125 (over the interval)

$$\frac{\Delta x}{1} = \frac{0.5}{3.125}$$

$$\Delta x = 0.16$$

Interval from x = 0 to x = 0.5: interval width 0.5, Δ x; P(x) changes by - 1 (to the root), - 1.375 (over the interval)

$$\frac{\Delta x}{1} = \frac{-0.5}{-1.375}$$

$$\Delta x = 0.36$$

Interval from x = 1.5 to x = 2: interval width 0.5, Δ x; P(x) changes by 0.125 (to the root), 3.125 (over the interval)

$$\frac{\Delta x}{0.125} = \frac{0.5}{3.125}$$

$$\Delta x = 0.02$$