

# Section 12-1

## Two Fundamental Counting Principles

## Objectives for Section 12-1

- to use the fundamental counting principles to find a total number of elements in a specified set.

## Two Fundamental Counting Principles

- If set  $A$  contains  $r$  elements, set  $B$  contains  $s$  elements, and  $A \cap B$  contains  $t$  elements, then  $A \cup B$  contains  $r + s - t$  elements.
- If set  $A$  contains  $r$  elements and set  $B$  contains  $s$  elements, then there are  $rs$  different ordered pairs  $(a, b)$  with  $a \in A$  and  $b \in B$  (that is,  $A \times B$  contains  $rs$  elements).

## Example of the First Fundamental Counting Principle

Out of a faculty of 76 at Central High School, 52 teach at least one eleventh-grade class, 45 teach at least one twelfth-grade class, and 3 teach no eleventh- or twelfth-grade classes. How many on the faculty teach both eleventh- and twelfth-grade classes?

If set A contains  $r$  elements, set B contains  $s$  elements, and  $A \cap B$  contains  $t$  elements, then  $A \cup B$  contains  $r + s - t$  elements.

$r = 52$ ;  $s = 45$ ;  $A \cap B$ ,  $t$ , is what you are solving for and  $A \cup B = 73$

$$52 + 45 - 73 = \mathbf{24}$$

## Example of the Second Fundamental Counting Principle

How many three-digit integers can be formed that do not contain a 4?

If set A contains  $r$  elements and set B contains  $s$  elements, then there are  $rs$  different ordered pairs  $(a, b)$  with  $a \in A$  and  $b \in B$  (that is,  $A \times B$  contains  $rs$  elements).

$$(8)(9)(9) = \mathbf{648}$$

# Section 12-2

## Linear and Circular Permutations

## Objectives for Section 12-2

- to calculate the number of linear and circular permutations of a given set

## Linear and Circular Permutations

- The number of permutations of the members of a set containing  $n$  different elements is  $n!$
- The number of permutations of  $r$  members of a set containing  $n$  different elements is
$${}_n P_r = n(n - 1)(n - 2) \dots (n - r + 1)$$

$${}_n P_r = \frac{n!}{(n - r)!}$$

- circular permutation:  $(n - 1)!$

## Example for a Linear Permutation of All Elements

How many arrangements are there of the letters in the word CURTAIN.

$$7! = \mathbf{5040}$$

## Example for a Linear Permutation of Some Elements

How many three letter permutations are there of the letters in the word COMPUTER.

$${}_8P_3 = \frac{8!}{(8-3)!} = 8 \cdot 7 \cdot 6 = \mathbf{336}$$

## Example for a Circular Permutation

In how many ways can 8 distinct dishes be arranged around a revolving platter at a buffet dinner?

$$(8 - 1)! = \mathbf{5040}$$

# Section 12-3

## Permutations with Repeated Elements

## Objectives for Section 12-3

- to calculate permutations of set with repeated elements

## Permutations with Repeated Elements

- The number of distinguishable permutations of  $n$  elements taken  $n$  at a time, with  $n_1$  elements alike,  $n_2$  of another kind alike, and so on, is

$$\frac{n!}{n_1!n_2!\dots}$$

## Example of Permutations with Repeated Elements

Find the number of permutations of the letters in the word, MILLIMETERS.

$n = 11$ ,  $n_1 = 2$  because there are two M's,  $n_2 = 2$  because there are two I's,  $n_3 = 2$  because there are L's,  $n_4 = 2$  because there are two E's

$$\frac{11!}{2!2!2!2!} = \mathbf{2,494,800}$$

# Section 12-4

## Counting Subsets

## Objectives for Section 12-4

- to count the number of combinations in a set

## Counting Subsets with Combinations

- combinations are subsets where order is not important
- to count the number of r-element subsets of a set containing n elements use

$${}_n C_r = \frac{n!}{r!(n-r)!}$$

## Example for Combinations

How many steering subcommittees of 6 people can be formed from a committee consisting of 10 people.

$${}_{10}C_6 = \frac{10!}{6!(10-6)!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = \mathbf{210}$$

# Section 12-5

## Combinations and Products

## Objectives for Section 12-5

- to use the fundamental counting principles in conjunction with combinations

## Combinations and Products

- Recall that when an event has multiple parts to it we count the choices for each part and then multiply those counts together to find the total number of possible events.
- You can use any method to find the total choices for each part including permutations and combinations. Remember to use a permutation when order counts and a combination when order does not count.

## Example for Product of Combinations

An urn contains 10 red marbles and 8 blue marbles.

In how many ways could you select 5 red and 7 blue marbles from the urn?

You have two parts to this event, red & blue.

$$({}_{10}C_5)({}_8C_7)$$

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# Section 12-6

## The Binomial Theorem

## Objectives for Section 12-6

- to expand a binomial using the Binomial Expansion Theorem
- to find a specified term in an expansion using the theorem
- to find the first four terms in an expansion using the theorem
- to use the theorem for expanding a binomial with a rational exponent

# Binomial Expansion

## The Binomial Theorem

- If  $n$  is a positive integer, then  $(a + b)^n =$

$$a^n + \frac{n!}{(n-1)!} a^{n-1} b + \frac{n!}{(n-2)!2!} a^{n-2} b^2 + \frac{n!}{(n-3)!3!} a^{n-3} b^3 + \dots + b^n$$

- Notice that the numerator of the coefficient is always the power of the binomial factorial.
- Notice that the denominator of the coefficient is always the product of the exponents of a factorial and the exponent of b factorial.
- Notice that the exponent of b is always one less than the position of the term in the polynomial.
- Notice that the exponents of a and b always have a sum equal to n, the power of the binomial.

## Example for 1-12

$$(x + 2y)^6$$

$$\frac{6!}{6!0!}(x)^6(2y)^0 + \frac{6!}{5!1!}(x)^5(2y)^1 + \frac{6!}{4!2!}(x)^4(2y)^2 + \frac{6!}{3!3!}(x)^3(2y)^3 + \frac{6!}{2!4!}(x)^2(2y)^4 + \frac{6!}{1!5!}(x)^1(2y)^5 + \frac{6!}{0!6!}(x)^0(2y)^6$$

$$1(x)^6(2y)^0 + 6(x)^5(2y)^1 + 15(x)^4(2y)^2 + 20(x)^3(2y)^3 + 15(x)^2(2y)^4 + 6(x)^1(2y)^5 + 1(x)^0(2y)^6$$

$$1(x)^6 + 12(x)^5(y)^1 + 60(x)^4(y)^2 + 160(x)^3(y)^3 + 240(x)^2(y)^4 + 192(x)^1(y)^5 + 64(y)^6$$

$$1x^6 + 12x^5y^1 + 60x^4y^2 + 160x^3y^3 + 240x^2y^4 + 192x^1y^5 + 64y^6$$

## Examples for 13-18

The eleventh term of  $(s - t)^{14}$

$$\frac{14!}{4!10!} s^4 (-t)^{10}$$

$$\frac{(14)(13)(12)(11)}{(4)(3)(2)} s^4 t^{10}$$

$$1001s^4 t^{10}$$

## Examples for 19-22

$$(a - b)^{14}$$

$$\frac{14!}{\binom{14}{0}}(a)(-b) + \frac{14!}{\binom{14}{1}}(a)(-b) + \frac{14!}{\binom{14}{2}}(a)(-b) + \frac{14!}{\binom{14}{3}}(a)(-b)$$

$$\frac{14!}{14!0!}(a)^{14}(-b)^0 + \frac{14!}{13!1!}(a)^{13}(-b)^1 + \frac{14!}{12!2!}(a)^{12}(-b)^2 + \frac{14!}{11!3!}(a)^{11}(-b)^3$$

$$\frac{14!}{14!0!}a^{14} - \frac{14!}{13!1!}a^{13}b + \frac{14!}{12!2!}a^{12}b^2 - \frac{14!}{11!3!}a^{11}b^3$$

$$1a^{14} - 14a^{13}b + \frac{(14)(13)}{2}a^{12}b^2 - \frac{(14)(13)(12)}{(3)(2)}a^{11}b^3$$

$$a^{14} - 14a^{13}b + 91a^{12}b^2 - 364a^{11}b^3$$

# Section 12-7

## Pascal's Triangle

## Objectives for Section 12-7

- to use Pascal's triangle to aid n binomial expansion.

# Binomial Expansion Using Pascal's Triangle

$$(a + b)^0 = \mathbf{1}$$

$$(a + b)^1 = \mathbf{1}a + \mathbf{1}b$$

$$(a + b)^2 = \mathbf{1}a^2 + \mathbf{2}ab + \mathbf{1}b^2$$

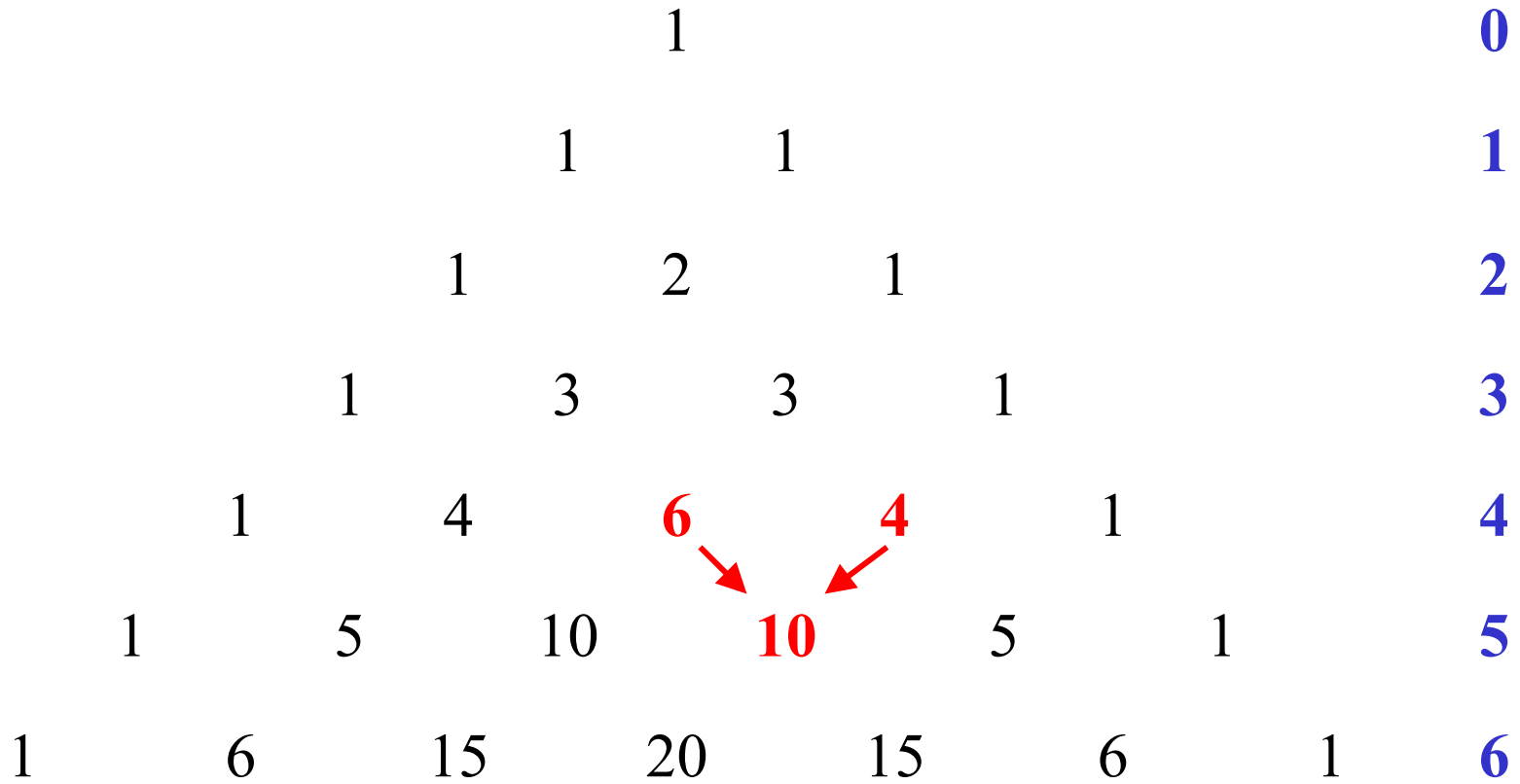
$$(a + b)^3 = \mathbf{1}a^3 + \mathbf{3}a^2b^1 + \mathbf{3}a^1b^2 + \mathbf{1}b^3$$

$$(a + b)^4 = \mathbf{1}a^4 + \mathbf{4}a^3b^1 + \mathbf{6}a^2b^2 + \mathbf{4}a^1b^3 + \mathbf{1}b^4$$

$$(a + b)^5 = \mathbf{1}a^5 + \mathbf{5}a^4b^1 + \mathbf{10}a^3b^2 + \mathbf{10}a^2b^3 + \mathbf{5}a^1b^4 + \mathbf{1}b^5$$

# Pascal's Triangle

**Row**



## Examples for 1-8

$$(x^2 - 1)^6$$

$$1(\ )(\ ) + 6(\ )(\ ) + 15(\ )(\ ) + 20(\ )(\ ) + 15(\ )(\ ) + 6(\ )(\ ) + 1(\ )(\ )$$

$$1(\ )^6(\ )^0 + 6(\ )^5(\ )^1 + 15(\ )^4(\ )^2 + 20(\ )^3(\ )^3 + 15(\ )^2(\ )^4 + 6(\ )^1(\ )^5 + 1(\ )^0(\ )^6$$

$$1(x^2)^6(-1)^0 + 6(x^2)^5(-1)^1 + 15(x^2)^4(-1)^2 + 20(x^2)^3(-1)^3 +$$

$$15(x^2)^2(-1)^4 + 6(x^2)^1(-1)^5 + 1(x^2)^0(-1)^6$$

$$\mathbf{1x^{12} - 6x^{10} + 15x^8 - 20x^6 + 15x^4 - 6x^2 + 1}$$

## Example for 9-14

$$(r^2 - t^3)^{14}; (t^3)^5$$

$$\frac{14!}{9!5!} (r^2)^9 (t^3)^5$$

$$\mathbf{- 2002r^{18}t^{15}}$$

# Section 12-8

## Sample Space and Events

## Objectives for Section 12-8

- to specify the sample space for an event
- to specify an event

## Sample Spaces and Events

- random experiment; an experiment where the same conditions do not necessarily result in the same outcome
- sample space: the set of all possible outcomes for a random experiment
- event: any subset of the sample space
- simple event: a subset with one element

## Example for 1-8

A tag is drawn from a box containing three tags numbered 1, 2 and 3. A tag is drawn and NOT returned to the box before a second tag is drawn.

Event: The sum of the two numbers is even.

**Sample Space:**  $\{(1, 2) (1, 3) (2, 1) (2, 3) (3, 1) (3, 2)\}$

**Event:**  $\{(1, 3) (3, 1)\}$

## Example for 9-16

Of two cards drawn from a standard 52 card deck, both are clubs.

Sample Space:  ${}_{52}C_2 = 1326$

Event:  ${}_{13}C_2 = 78$

# Section 12-9

## Mathematical Probability

## Objectives for 12-9

- to calculate the mathematical probability of a specified event

# Mathematical Probability

- Let  $S$  be the sample space of an experiment in which there are  $n$  possible outcomes, each equally likely. If an event  $E$  with  $h$  elements is a subset of  $S$ , then the probability of  $E$ , denoted by  $P(E)$  is given by,

$$P(E) = \frac{h}{n}$$

- the probability of an event where  $h = n$  is one and the event is called certain
- the probability of an event where  $h = 0$ , or  $E$  is  $\phi$ , is 0 and the event is called impossible
- the probability of any event that is not certain or impossible must be between 0 and 1
- complement of event  $E$  ( $\bar{E}$ ) is the subset of the sample space  $S$  that contains all of the elements that are not in  $E$
- $P(\bar{E})$  is the probability that  $E$  will NOT occur

## Example for 1-15

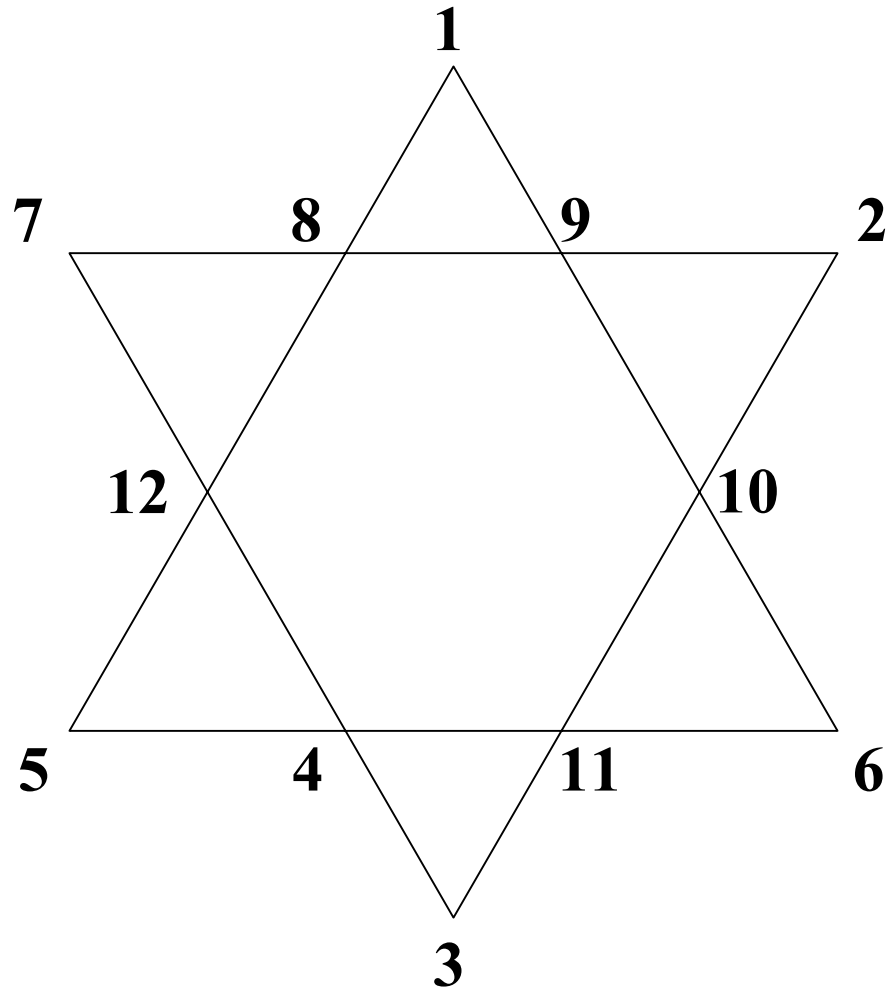
Four cards are drawn at random from the 13 hearts in a standard deck. What is the probability that the selection contains

$$\text{no face cards: } \frac{{}_{10}C_4}{{}_{13}C_4} = \frac{42}{143}$$

$$\text{both the king and the queen: } \frac{{}_{11}C_2}{{}_{13}C_4} = \frac{1}{13}$$

$$\text{the queen and not the king: } \frac{{}_{11}C_3}{{}_{13}C_4} = \frac{3}{13}$$

$$\text{at least two face cards: } \frac{({}_3C_2)({}_{10}C_2)}{{}_{13}C_4} + \frac{({}_3C_3)({}_{10}C_1)}{{}_{13}C_4} = \frac{27}{143} + \frac{2}{143} = \frac{29}{143}$$



# Section 12-10

Mutually Exclusive Events

## Objectives for Section 12-10

- to calculate the probability of the union of events some of which are mutually exclusive

## Mutually Exclusive Events

- mutually exclusive events: for any two events A and B,  
 $A \cap B = \phi$
- For any two events A and B in a sample space,  
 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- If A and B are mutually exclusive events,  
 $P(A \cup B) = P(A) + P(B)$

## Example for 1-8

Two cards are drawn from a standard 52 card deck. What is the probability that the cards are:

$$\text{both spades: } \frac{{}^{13}C_2}{{}^{52}C_2} = \frac{1}{17} \quad \text{of the same suit: } 4 \left( \frac{{}^{13}C_2}{{}^{52}C_2} \right) = \frac{4}{17}$$

$$\text{both jacks: } \frac{{}^4C_2}{{}^{52}C_2} = \frac{1}{221} \quad \text{a pair: } 13 \left( \frac{{}^4C_2}{{}^{52}C_2} \right) = \frac{1}{17}$$

$$\text{both spades or both jacks: } \frac{1}{221} + \frac{1}{17} = \frac{14}{221}$$

$$\text{both of the same suit or a pair: } \frac{4}{17} + \frac{1}{17} = \frac{5}{17}$$

# Section 12-11

## Conditional Probability

## Objectives for Section 12-11

- to calculate conditional probability

# Conditional Probability

- conditional probability: Let  $P(A)$  denote the probability of an event  $A$ , and  $P(B \mid A)$  denote the conditional probability of an event  $B$  given that event  $A$  has occurred. If  $P(A \cap B)$  is the probability that  $A$  and  $B$  occur, then
$$P(A \cap B) = P(A) \cdot P(B \mid A)$$
- independent events: when the probability of one does not depend on the occurrence of another. Two events  $A$  and  $B$  are independent if and only if  $P(A \cap B) = P(A) \cdot P(B)$ . For independent events  $P(A \mid B) = P(A)$  and  $P(B \mid A) = P(B)$
- dependent events: when the probability of one depends on the occurrence of the other.

## Example for 1-14

A single die is rolled twice. If A is the event that the sum of the numbers showing is 7, and B is the event of at least one 3 showing, calculate  $P(A)$ ,  $P(B)$ ,  $P(A | B)$ , and  $P(B | A)$  and show that  $P(A) \cdot P(B | A) = P(B) \cdot P(A | B)$

$$P(A) = \frac{1}{6} \quad P(B) = \frac{11}{36} \quad P(A \cap B) = \frac{2}{36} = \frac{1}{18}$$

$$P(A \cap B) = P(A) \cdot P(B | A) \quad P(A \cap B) = P(B) \cdot P(A | B)$$

$$\frac{1}{18} = \frac{1}{6} \cdot P(B|A)$$

$$\frac{1}{18} = \frac{11}{36} \cdot P(A|B)$$

$$\frac{1}{3} = P(B|A)$$

$$\frac{2}{11} = P(A|B)$$

$$\frac{1}{6} \cdot \frac{1}{3} = \frac{11}{36} \cdot \frac{2}{11}$$

