Section 13-1

Matrices and Their Sums

Objectives for 13-1

- to find the sum or difference of matrices
- to solve simple matrix equations involving only sums and differences

Matrices

- matrix: a rectangular array of numbers enclosed with brackets, named with a capital letter
- elements (entry): the numbers in a matrix
- index of an element (entry): an ordered pair (row, column) of numbers representing the position of the element in the array
- dimensions: (number of rows) x (number of columns)
- row matrix: an array with a single row
- column matrix: an array with a single column
- square matrix: an array with equal numbers of rows and columns or an (n x n) matrix
- zero matrix ($O_{m \times n}$): every element in the array is zero
- equal matrices: two matrices are equal if and only if they have the same dimensions and the elements in all corresponding positions are equal.

Example for 1-8

$$\begin{bmatrix} 12 & 7 & -5 \\ -1 & 0 & 4 \\ 3 & 10 & 11 \end{bmatrix} + \begin{bmatrix} 8 & 15 & -6 \\ 5 & -2 & -7 \\ 7 & -10 & 5 \end{bmatrix}$$

$$\begin{bmatrix} (12+8) & (7+15) & (-5-6) \\ (-1+5) & (0-2) & (4-7) \\ (3+7) & (10-10) & (11+5) \end{bmatrix}$$

$$\begin{bmatrix} 20 & 22 & -11 \\ 4 & -2 & -3 \\ 10 & 0 & 16 \end{bmatrix}$$

Example for 9-16

$$\begin{bmatrix} w+3 & x \\ y-4 & z \end{bmatrix} + \begin{bmatrix} x-2 & -z \\ 3 & 3z \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} (w+3+x-2) & (x-z) \\ (y-4+3) & (z+3z) \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 1 & -2 \end{bmatrix}$$

$$x + w + 1 = 5$$

$$x - z = 2$$

$$y - 1 = 1 \qquad \mathbf{y = 2}$$

$$4z = -2 \qquad \mathbf{z = -\tfrac{1}{2}}$$

$$x = \tfrac{3}{2}$$

$$w = \tfrac{5}{2}$$

Section 13-2

Properties of Matrix Addition

Objectives for Section 13-2

- to solve a matrix equation for a variable matrix
- to prove the properties of matrices

Arithmetic Properties of Matrices

Let A, B and C be m x n matrices. Let $O_{m \times n}$ be the m x n zero matrix.

- Closure Property: A + B is an m x n matrix
- Commutative Property: A + B = B + A
- Associative Property: (A + B) + C = A + (B + C)
- Identity Property: $A + O_{m \times n} = O_{m \times n} + A = A$
- Inverse Property: $A + (-A) = -A + A = O_{m \times n}$

Example for 1-8

$$\begin{bmatrix} 6 & -10 & 0 \\ -9 & 7 & 4 \end{bmatrix} - X = \begin{bmatrix} 4 & -6 & 2 \\ 1 & 0 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -10 & 0 \\ -9 & 7 & 4 \end{bmatrix} - \begin{bmatrix} 4 & -6 & 2 \\ 1 & 0 & -3 \end{bmatrix} = X$$

$$\begin{bmatrix} 2 & -4 & -2 \\ -10 & 7 & 7 \end{bmatrix} = X$$

Examples for 9&10

$$\begin{bmatrix} a \\ b \end{bmatrix} + X = \begin{bmatrix} c \\ d \end{bmatrix}$$

$$X = \begin{bmatrix} c \\ d \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix}$$

$$X = \begin{bmatrix} c - a \\ d - b \end{bmatrix}$$

Example for 11-16

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a-a & b-b \\ c-c & d-d \end{bmatrix}$$

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Section 13-3

Product of a Matrix and a Scalar

Objectives for Section 13-3

- to find sums, differences and scalar products of matrices
- to solve matrix equations that involve sums, differences and scalar products
- to prove statements about scalar products of matrices

Properties of Scalar Multiplication

Let A, B and C be m x n matrices. Let $O_{m \times n}$ be the m x n zero matrix, and let p and q be scalars.

- Closure Property: pA is an m x n matrix
- Commutative Property: pA = Ap
- Associative Property: p(qA) = (pq)A
- Distributive Property: (p + q)A = pA + qA
  p(A+ B) = pA + pB
- Identity Property: 1(A) = A
- Multiplicative Property of - 1: (- 1)A = - A
- Multiplicative Property of 0: $0(A) = O_{m \times n}$

Example for 1-18

$$2X - 3A = B$$

$$A = \begin{bmatrix} 9 & -3 \\ -21 & 0 \end{bmatrix}, B = \begin{bmatrix} 11 & -7 \\ 1 & 10 \end{bmatrix}$$

$$2X - 3\begin{bmatrix} 9 & -3 \\ -21 & 0 \end{bmatrix} = \begin{bmatrix} 11 & -7 \\ 1 & 10 \end{bmatrix}$$

$$2X = \begin{bmatrix} 11 & -7 \\ 1 & 10 \end{bmatrix} + 3\begin{bmatrix} 9 & -3 \\ -21 & 0 \end{bmatrix}$$

$$2X = \begin{bmatrix} 11 & -7 \\ 1 & 10 \end{bmatrix} + \begin{bmatrix} 27 & -9 \\ -63 & 0 \end{bmatrix}$$

$$2X = \begin{bmatrix} 38 & -16 \\ -62 & 10 \end{bmatrix}$$

$$X = \frac{1}{2}\begin{bmatrix} 38 & -16 \\ -62 & 10 \end{bmatrix}$$

$$X = \begin{bmatrix} 19 & -8 \\ -31 & 5 \end{bmatrix}$$

Example for 19-27

$$A = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

$$(c + d)A = cA + dA$$

$$(c+d)\begin{bmatrix} w & x \\ y & z \end{bmatrix} = c\begin{bmatrix} w & x \\ y & z \end{bmatrix} + d\begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

$$(c+d)\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} cw & cx \\ cy & cz \end{bmatrix} + \begin{bmatrix} dw & dx \\ dy & dz \end{bmatrix}$$

$$(c+d)\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} cw+dw & cx+dx \\ cy+dy & cz+dz \end{bmatrix}$$

$$(c+d)\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} (c+d)w & (c+d)x \\ (c+d)y & (c+d)z \end{bmatrix}$$

Section 13-4

Product of Two Matrices

Objectives for 13-4

- to find the product, if any, between two matrices
- to prove basic properties of matrix products
- to solve matrix equations using multiplication

Matrix Multiplication

- The product of matrices $A_{m \times n}$ and $B_{n \times p}$ is the m x p matrix whose element in the a^{th} row b^{th} column is the sum of the products of corresponding elements of the a^{th} row of A and the b^{th} column of B.
- Two matrices can be multiplied only if the number of columns in the first matrix is equal to the number of rows in the second matrix.
- identity matrix($I_{n \times n}$): a square matrix whose main diagonal, from element (1,1) to (n, n), has all 1's and all other elements are 0.

Example for 1-12

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 1 & 0 \end{bmatrix}$$

A 3x2 multiplied to a 2x4 creates a 3x4 matrix.

$$\begin{bmatrix} e_{1,1} & e_{1,2} & e_{1,3} & e_{1,4} \\ e_{2,1} & e_{2,2} & e_{2,3} & e_{2,4} \\ e_{3,1} & e_{3,2} & e_{3,3} & e_{3,4} \end{bmatrix} = \begin{bmatrix} -1 & 2 & 2 & -1 \\ 4 & -1 & -1 & -3 \\ -4 & 4 & 4 & 0 \end{bmatrix}$$

$e_{1,1} = (1)(1) + (2)(-1) = -1$ $e_{2,1} = (3)(1) + (-1)(-1) = 4$ $e_{3,1} = (0)(1) + (4)(-1) = -4$

$e_{1,2} = (1)(0) + (2)(1) = 2$ $e_{2,2} = (3)(0) + (-1)(1) = -1$ $e_{3,2} = (0)(0) + (4)(1) = 4$

$e_{1,3} = (1)(0) + (2)(1) = 2$ $e_{2,3} = (3)(0) + (-1)(1) = -1$ $e_{3,3} = (0)(0) + (4)(1) = 4$

$e_{1,4} = (1)(-1) + (2)(0) = -1$ $e_{2,4} = (3)(-1) + (-1)(0) = -3$ $e_{3,4} = (0)(-1) + (4)(0) = 0$

Example for 13-18

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix},\ B = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix},\ C = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

Find (A + B)(A - B) and $A^2 - B^2$.

$$\left(\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\right)\left(\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\right)$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\right) - \left(\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\right)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

Notice $(A + B)(A - B) \neq A^2 - B^2$.

Example for 21-26

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 5 \end{bmatrix}$$

$$2x + 0y - 1z = 6$$

$$0x + 3y + 0z = 9$$

$$1x + 1y + 0z = 5$$

$$\mathbf{y = 3}$$

$$x + 3 = 5$$

$$\mathbf{x = 2}$$

$$4 - z = 6$$

$$\mathbf{z = -2}$$

Section 13-5

Properties of Matrix Multiplication

Objectives for Section 13-5

- to compute and compare matrix products
- to prove basic properties of matrix products

Properties of Matrix Multiplication

Let A, B and C be n x n matrices. Let $I_{n \times n}$ be the identity matrix and $O_{n \times n}$ be the zero matrix.

- Closure Property: AB is an n x n matrix
- Associative Property: $(AB)C = A(BC)$
- Distributive Property: $A(B + C) = AB + AC$
  $(B + C)A = BA + CA$
- Identity Property: $I_{n \times n} \bullet A = A \bullet I_{n \times n} = A$
- Multiplicative Property of $O_{n \times n}$: $O_{n \times n} \bullet A = A \bullet O_{n \times n} = O_{n \times n}$
- Associative Property for Scalars: $a(AB) = (aA)B = A(aB)$
- **There is no Commutative Property for matrix multiplication. If you change the order you change the answer.**

Example for 1-14

$$A=\begin{bmatrix}4 & 2\\ -8 & -4\end{bmatrix}, B=\begin{bmatrix}-1 & 3\\ 2 & -6\end{bmatrix}, C=\begin{bmatrix}8 & 2\\ 4 & 6\end{bmatrix}, D=\begin{bmatrix}3 & -1\\ -2 & 4\end{bmatrix}$$

Compare BD and DB

$$\begin{bmatrix}-1 & 3\\ 2 & -6\end{bmatrix}\begin{bmatrix}3 & -1\\ -2 & 4\end{bmatrix} \qquad \begin{bmatrix}3 & -1\\ -2 & 4\end{bmatrix}\begin{bmatrix}-1 & 3\\ 2 & -6\end{bmatrix}$$

$$\begin{bmatrix}-9 & 13\\ 18 & -26\end{bmatrix} \neq \begin{bmatrix}-5 & 15\\ 10 & -30\end{bmatrix}$$

Example for 15-18

a. $(A - B)(C + D)$
b. $AC - BC - BD + AD$
c. $A(C + D) - B(C + D)$

Keep in mind that matrix multiplication is not commutative.
Throw out any that require the commutative property to be true.

a, b and c are equal

Section 13-6

Matrix Solution of a Linear System

Objectives for Section 13-6

- to find the inverse of a 2 x 2 matrix
- to determine if a matrix is singular
- to solve a matrix equation
- to solve a system with a matrix equation

Inverse of a Matrix

- inverse matrices: for any two matrices A and B, if $AB = BA = I$, then A and B are inverse matrices and $B = A^{-1}$ and $A = B^{-1}$.
- Inverses of a 2x2 Matrix:

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } \det A \neq 0, \text{ then } A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $\det A = 0$, then A has no inverse and is called singular.

Example for 1-10

$$A = \begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 3 & -1 \\ 6 & 2 \end{vmatrix} = (3)(2) - (6)(-1) = 6 + 6 = 12$$

$$A^{-1} = \frac{1}{12}\begin{bmatrix} 2 & 1 \\ -6 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{2}{12} & \frac{1}{12} \\ \frac{-6}{12} & \frac{3}{12} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{6} & \frac{1}{12} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

Example for 11-18

$$\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} X - \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} X - \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{3}{2} \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{3}{2} \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} X = \begin{bmatrix} 2 & \frac{7}{2} \\ -1 & \frac{5}{2} \end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} X = \frac{1}{2}\begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 & \frac{7}{2} \\ -1 & \frac{5}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X = \frac{1}{2}\begin{bmatrix} 4 & -\frac{3}{2} \\ -6 & \frac{13}{2} \end{bmatrix} \qquad X = \begin{bmatrix} 2 & -\frac{3}{4} \\ -3 & \frac{13}{4} \end{bmatrix}$$

Example for 19-24

2x - y = 6

3x + 2y = - 19

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -19 \end{bmatrix}$$

$$\frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -19 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -7 \\ -56 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ -8 \end{bmatrix}$$

(- 1, -8)

Section 13-7

Transformations by Matrix Addition

Objectives for Section 13-7

- to find the image of a point under a matrix translation
- to find the preimage of a point under a matrix translation
- to identify the vector matrix associated with a matrix translation and then to use that vector matrix to find the image of a point
- to identify the vector matrix associated with a matrix translation and then to use that vector matrix to find the preimage of a point

Transformations by Matrix Addition

- preimage: the original or starting position
- image: the copy or final position
- mapping: a correspondence between a set of points such that for any value of the preimage there exists only one image, written
  - Name: (coordinates of preimage) $\rightarrow$ (coordinates of the image)
- relation: correspondence between sets of numbers
- function: correspondence between sets of numbers such that for any value of the domain there exists only one value of the range. Written as:
  - equation: $y = mx + b$
  - function notation: $f(x) = mx + b$
  - mapping notation: $f{:}x \rightarrow mx + b$

Transformations by Matrix Addition

- one-to-one mapping: every value of the image has exactly one preimage.
- transformation: one-to-one mapping of the whole plane
- isometry (congruence mapping): a transformation that preserves distance
- translation: a transformation that moves all points the same vector (h, v).
  - mapping notation $T:(x, y) \rightarrow (x + h, y + v)$
  - matrix notation: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ v \end{bmatrix}$

Sample for 1-6

$$P(a, a-b); \begin{bmatrix} a \\ a+b \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ v \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a \\ a-b \end{bmatrix} + \begin{bmatrix} a \\ a+b \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2a \\ 2a \end{bmatrix}$$

P'(2a, 2a)

Sample for 7-12

$$P'(a,-b);\ \begin{bmatrix} a - b \\ a + b \end{bmatrix}$$

$$\begin{bmatrix} a \\ -b \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a-b \\ a+b \end{bmatrix}$$

$$\begin{bmatrix} a \\ -b \end{bmatrix} - \begin{bmatrix} a-b \\ a+b \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} b \\ -a-2b \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

P(b, - a – 2b)

Sample for 13-18

P(a – b, a + b) P'(a, b)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ v \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a - b \\ a + b \end{bmatrix} + \begin{bmatrix} h \\ v \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} a - b \\ a + b \end{bmatrix} = \begin{bmatrix} h \\ v \end{bmatrix}$$

$$\begin{bmatrix} b \\ -a \end{bmatrix} = \begin{bmatrix} h \\ v \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b \\ -a \end{bmatrix}$$

Sample for 19-22

X(a, b); P(c, d); P'(- a, b)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ v \end{bmatrix}$$

$$\begin{bmatrix} -a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} + \begin{bmatrix} h \\ v \end{bmatrix}$$

$$\begin{bmatrix} -a \\ b \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} h \\ v \end{bmatrix}$$

$$\begin{bmatrix} -a-c \\ b-d \end{bmatrix} = \begin{bmatrix} h \\ v \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} -a-c \\ b-d \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -c \\ 2b-d \end{bmatrix}$$

X'(- c, 2b – d)

Sample for 23-26

X'(a, b); P(c, d); P'(2a, b)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ v \end{bmatrix}$$

$$\begin{bmatrix} 2a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} + \begin{bmatrix} h \\ v \end{bmatrix}$$

$$\begin{bmatrix} 2a \\ b \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} h \\ v \end{bmatrix}$$

$$\begin{bmatrix} 2a - c \\ b - d \end{bmatrix} = \begin{bmatrix} h \\ v \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2a - c \\ b - d \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} 2a - c \\ b - d \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} c - a \\ d \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

X(c - a, d)

Section 13-8

Transformations by Matrix Multiplication

Objectives for Section 13-8

- to find the image of a point under a linear transformation
- to find the preimage of a point under a linear transformation
- to find the equations for x’ and y’ under a linear transformation and to find the slope of the line for all points P’
- to describe the mapping of the plane onto itself under a linear transformation

Transformations by Matrix Multiplication

- linear transformation of the plane: is any equation of the form X' = AX where a_1, b_1, a_2 and $b_2 \in \Re$ and

$$X' = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix}$$

- If det A $\neq$ 0 then the transformation is nonsingular and is a one-to-one mapping of the plane onto itself.
- If A = O then the transformation maps the entire plane onto the origin.
- If A $\neq$ O but det A = 0 then the transformation maps the entire plane onto a line through the origin of the form $y' = mx'$

Example for 1-6

$$P(4,3); \begin{bmatrix} -1 & 2 \\ 5 & -3 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$$

P’(2, 11)

Example for 7-12

$$P'(4,-3);\ \begin{bmatrix} 2 & 5 \\ 3 & 9 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{1}{3} \begin{bmatrix} 9 & -5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 17 \\ -6 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

P(17, - 6)

Example for 13-16

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 6 & 15 \\ 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = 6x + 15y$$

$$y' = 4x + 10y$$

$$y' = mx'$$

$$4x + 10y = m(6x + 15y)$$

$$2(2x + 5y) = 3m(2x + 5y)$$

$$\frac{2(2x + 5y)}{3(2x + 5y)} = m$$

$$\frac{2}{3} = m$$

Example for 17-20

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

A parallelogram with vertices (0, 0), (1, 2), (1, 3) and (0, 1).

Example for 21-28

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ -y \end{bmatrix}$$

(x', y') = (0, - y) A projection onto the y-axis followed by a reflection into the x-axis.