Section 15-1

Fundamental Identities

Objectives for Section 15-1

- to write an equivalent trigonometric expression using only a single function

Reciprocal Identities

$$\sin\alpha = \frac{1}{\csc\alpha} \qquad \sin\alpha\csc\alpha = 1 \qquad \csc\alpha = \frac{1}{\sin\alpha}$$

$$\cos\alpha = \frac{1}{\sec\alpha} \qquad \cos\alpha\sec\alpha = 1 \qquad \sec\alpha = \frac{1}{\cos\alpha}$$

$$\tan\alpha = \begin{cases} \dfrac{1}{\cot\alpha} \\ \dfrac{\sin\alpha}{\cos\alpha} \end{cases} \qquad \tan\alpha\cot\alpha = 1 \qquad \cot\alpha = \begin{cases} \dfrac{1}{\tan\alpha} \\ \dfrac{\cos\alpha}{\sin\alpha} \end{cases}$$

Cofunction Identities

$\sin\theta = \cos(90° - \theta)$
$\tan\theta = \cot(90° - \theta)$
$\sec\theta = \csc(90° - \theta)$

$\cos\theta = \sin(90° - \theta)$
$\cot\theta = \tan(90° - \theta)$
$\csc\theta = \sec(90° - \theta)$

Pythagorean Identities

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$1 + \tan^2\alpha = \sec^2\alpha$$

$$1 + \cot^2\alpha = \csc^2\alpha$$

General Strategies for Proving Identities

- Simplify the more complicated side of the identity until it is identical to the other side.
- Transform both sides of the identity into the same expression.

Special Strategies for Proving Identities

- Express functions in terms of sines and cosines.
- Look for expressions to which the Pythagorean identities can be applied.
- Use factoring.
- Combine terms on each side of the identity into a single fraction.
- Multiply one side of the equation by an expression equal to one.

Examples for 1-32

$$\tan \alpha \sec \alpha$$

$$\left(\frac{\sin\alpha}{\cos\alpha}\right)\left(\frac{1}{\cos\alpha}\right)$$

$$\frac{\sin\alpha}{\cos^2\alpha}$$

$$\frac{\sin\alpha}{1-\sin^2\alpha};\cos\alpha \neq 0;\sin^2\alpha \neq 1$$

Section 15-2

Proving Identities

Objectives for Section 15-2

- to prove trigonometric identities

General Strategies for Proving Identities

- Simplify the more complicated side of the identity until it is identical to the other side.
- Transform both sides of the identity into the same expression.

Special Strategies for Proving Identities

- Express functions in terms of sines and cosines.
- Look for expressions to which the Pythagorean identities can be applied.
- Use factoring.
- Combine terms on each side of the identity into a single fraction.
- Multiply one side of the equation by an expression equal to one.

Examples for 1-28

$\cot\alpha + \tan\alpha = 2\csc 2\alpha$

$$2\left(\frac{1}{\sin 2\alpha}\right)$$

$$2\left(\frac{1}{2\sin\alpha\cos\alpha}\right)$$

$$\frac{1}{\sin\alpha\cos\alpha}$$

$$\frac{\sin^2\alpha + \cos^2\alpha}{\sin\alpha\cos\alpha}$$

$$\frac{\sin^2\alpha}{\sin\alpha\cos\alpha} + \frac{\cos^2\alpha}{\sin\alpha\cos\alpha}$$

$$\frac{\sin\alpha}{\cos\alpha} + \frac{\cos\alpha}{\sin\alpha}$$

$$\boldsymbol{\cot\alpha + \tan\alpha}$$

Section 15-3

The Cosine of a Sum or Difference

Objectives for Section 15-3

- to recognize the formulas for cosine angle sums and differences in order to find the exact value of a trig function
- to use cosine sums and differences to find the exact value of non-pattern angles
- to express non-pattern angles as a cosine of a sum or difference in both degrees and radians
- to use cosine sum and difference formulas to simplify trig statements
- to find the cosine sums and differences of two angles when given one of their trig function values

Addition Formulas for Cosine

- $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$
- $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$

Example for 1-8

$$\cos 175°\cos 25° + \sin 175°\sin 25°$$

$$\cos(175° - 25°)$$

$$\cos 150°$$

$$-\frac{\sqrt{3}}{2}$$

Examples for 9-20

$$\cos 105°$$

$$\cos (60° + 45°)$$

$$\cos 60° \cos 45° - \sin 60° \sin 45°$$

$$\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$\frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4}$$

$$\frac{\sqrt{2}-\sqrt{6}}{4}$$

Examples for 21-28

$$\cos\left(\frac{\pi}{3}+\theta\right)+\cos\left(\frac{\pi}{3}-\theta\right)=\cos\theta$$

$$\cos\frac{\pi}{3}\cos\theta-\sin\frac{\pi}{3}\sin\theta+\cos\frac{\pi}{3}\cos\theta+\sin\frac{\pi}{3}\sin\theta=$$

$$2\cos\frac{\pi}{3}\cos\theta=$$

$$2\left(\frac{1}{2}\right)\cos\theta=$$

$$\cos\theta=$$

Examples for 29-36

$$\sin\alpha = -\frac{3}{5}; \pi < \alpha < \frac{3\pi}{2} \quad \cos\beta = \frac{8}{17}; \frac{3\pi}{2} < \beta < 2\pi$$

$$\cos\alpha = -\frac{4}{5}; \sin\beta = -\frac{15}{17}$$

$$\cos(\alpha+\beta) = \left(-\frac{4}{5}\right)\left(\frac{8}{17}\right) - \left(-\frac{3}{5}\right)\left(-\frac{15}{17}\right)$$

$$\cos(\alpha+\beta) = -\frac{77}{85}$$

Section 15-4

The Sine and Tangent of a Sum or Difference

Objectives for Section 15-4

- to recognize the formulas for sine and tangent angle sums and differences in order to find the exact value of a trig function
- to use sine and tangent sums and differences to find the exact value of non-pattern angles
- to express non-pattern angles as a sine and tangent of a sum or difference in both degrees and radians
- to use sine and tangent sum and difference formulas to simplify trig statements
- to find the sine and tangent sums and differences of two angles when given one of their trig function values

Addition Formulas for Sine and Tangent

- $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
- $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
- $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$
- $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

Example for 1-10

$$\sin 70° \cos 170° + \cos 70° \sin 170°$$

$$\sin(70° + 170°)$$

$$\sin(240°)$$

$$-\frac{\sqrt{3}}{2}$$

Example for 11-26, 39-46

$$\sin 165°$$

$$\sin(120° + 45°)$$

$$\sin 120° \cos 45° + \cos 120° \sin 45°$$

$$\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$\frac{\sqrt{6} - \sqrt{2}}{4}$$

Example for 27-38

$$\sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta$$

$$\sin\frac{\pi}{2}\cos\theta+\cos\frac{\pi}{2}\sin\theta$$

(1)(cos θ) + (0)(sin θ)

cos θ

Example for 47-54

$$\sin\alpha = -\frac{3}{5}; \pi < \alpha < \frac{3\pi}{2} \quad \cos\beta = \frac{8}{17}; \frac{3\pi}{2} < \beta < 2\pi$$

$$\cos\alpha = -\frac{4}{5}; \sin\beta = -\frac{15}{17}$$

$$\sin(\alpha+\beta) = \left(-\frac{3}{5}\right)\left(\frac{8}{17}\right) + \left(-\frac{4}{5}\right)\left(-\frac{15}{17}\right)$$

$$\sin(\alpha+\beta) = \frac{36}{85}$$

Section 15-5

Double-Angle and Half-Angle Formulas

Objectives for Section 15-5

- to find the exact values of the double and half angle values for angles given information about their quadrant and the value of one of their trigonometric functions
- to find the exact value of and angle by writing it as either half or double of a pattern angle
- to use half and double angle formulas to prove trigonometric identities

Double-Angle Formulas for Sine, Cosine and Tangent

- $\sin 2\alpha = 2\sin \alpha \cos \alpha$
- $\cos 2\alpha = \cos^2 \alpha - \sin^2\alpha$
- $\cos 2\alpha = 1 – 2\sin^2\alpha$
- $\cos 2\alpha = 2\cos^2\alpha - 1$
- $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

Half-Angle Formulas for Sine, Cosine and Tangent

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$$

$$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$$

Example for 1-6

$$\text{III}; \sin\alpha = -\frac{3}{5}$$

$$\cos\alpha = -\frac{4}{5}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\sin 2\alpha = 2\left(-\frac{3}{5}\right)\left(-\frac{4}{5}\right)$$

$$\frac{24}{25}$$

Example for 7-14

$$\text{II}; \sin\alpha = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

$$\cos 2\alpha = 1 - 2\sin^2\alpha$$

$$\cos 2\alpha = 1 - 2\left(\frac{2}{\sqrt{5}}\right)^2$$

$$\cos 2\alpha = 1 - \frac{8}{5}$$

$$-\frac{3}{5}$$

Example for 15-26

$$\sin 22.5^\circ$$

$$\sin 22.5^\circ = \sin \frac{45^\circ}{2} = +\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}$$

$$+\sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}}$$

$$+\sqrt{\frac{2-\sqrt{2}}{4}}$$

$$+\frac{\sqrt{2-\sqrt{2}}}{2}$$

Example for 27-32

$$\cos\alpha = -\frac{7}{9}, 90^\circ < m(\alpha) < 180^\circ$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \qquad \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\left(-\frac{7}{9}\right)}{2}} \qquad \cos\frac{\theta}{2} = \sqrt{\frac{1+\left(-\frac{7}{9}\right)}{2}}$$

$$\sin\frac{\theta}{2} = \sqrt{\frac{\frac{16}{9}}{2}} \qquad \cos\frac{\theta}{2} = \sqrt{\frac{\frac{2}{9}}{2}}$$

$$\sin\frac{\theta}{2} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3} \qquad \cos\frac{\theta}{2} = \sqrt{\frac{1}{9}} = \frac{1}{3}$$

Section 15-6

More on Identities

Objectives for Section 15-6

- to use all of the formulas presented to date to solve trigonometric identities

Reciprocal Identities

$$\sin\alpha = \frac{1}{\csc\alpha} \qquad \sin\alpha\csc\alpha = 1 \qquad \csc\alpha = \frac{1}{\sin\alpha}$$

$$\cos\alpha = \frac{1}{\sec\alpha} \qquad \cos\alpha\sec\alpha = 1 \qquad \sec\alpha = \frac{1}{\cos\alpha}$$

$$\tan\alpha = \begin{cases} \dfrac{1}{\cot\alpha} \\ \dfrac{\sin\alpha}{\cos\alpha} \end{cases} \qquad \tan\alpha\cot\alpha = 1 \qquad \cot\alpha = \begin{cases} \dfrac{1}{\tan\alpha} \\ \dfrac{\cos\alpha}{\sin\alpha} \end{cases}$$

Cofunction Identities

$\sin\theta = \cos(90° - \theta)$

$\tan\theta = \cot(90° - \theta)$

$\sec\theta = \csc(90° - \theta)$

$\cos\theta = \sin(90° - \theta)$

$\cot\theta = \tan(90° - \theta)$

$\csc\theta = \sec(90° - \theta)$

Opposite Identities

- $\sin(-x) = -\sin x$
- $\cos(-x) = \cos x$
- $\tan(-x) = -\tan x$

Pythagorean Identities

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$1 + \tan^2\alpha = \sec^2\alpha$$

$$1 + \cot^2\alpha = \csc^2\alpha$$

Addition and Subtraction Formulas

- $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$
- $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$
- $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$
- $\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$
- $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$
- $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}$

Double-Angle Formulas

- $\sin 2\alpha = 2\sin \alpha \cos \alpha$
- $\cos 2\alpha = \cos^2 \alpha - \sin^2\alpha$
- $\cos 2\alpha = 1 – 2\sin^2\alpha$
- $\cos 2\alpha = 2\cos^2\alpha - 1$
- $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

Half-Angle Formulas

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$$

$$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$$

Section 15-7

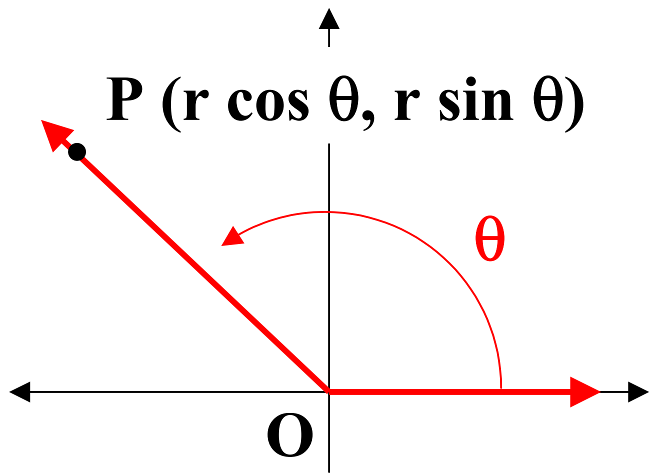
The Law of Cosines

Objectives for 15-7

- to find missing lengths in a triangle using the law of cosines
- to find the measure of missing angles in a triangle using the law of cosines
- to solve triangles for all missing angles and sides using the law of cosines

Law of Cosines

If θ is an angle in standard position and P is a point on its terminal side, then the coordinates of P are (r cos θ, r sin θ) where r = OP.

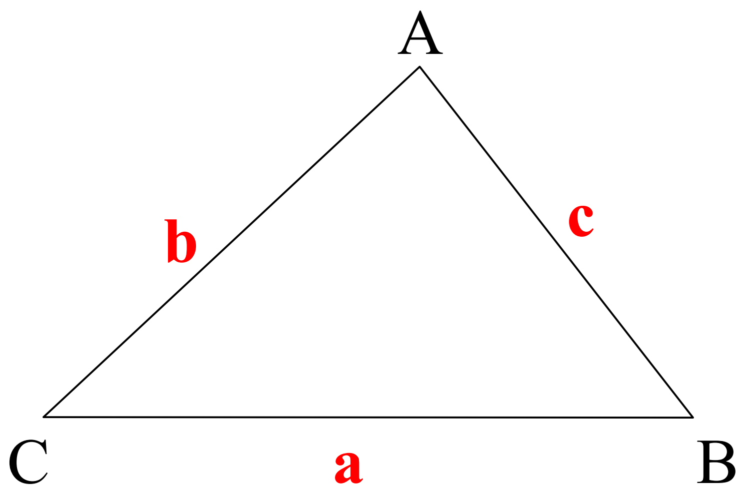


In any triangle ABC,

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$



Examples for 1-16

$$a = 6,\ b = 7,\ \angle C = 20°,\ c = ?$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 6^2 + 7^2 - 2(6)(7)(\cos 20)$$

$$c^2 = 36 + 49 - (84)(0.9397)$$

$$c^2 = 6.0652$$

$$c = \sqrt{6.0652} = 2.46$$

Section 15-8

The Law of Sines

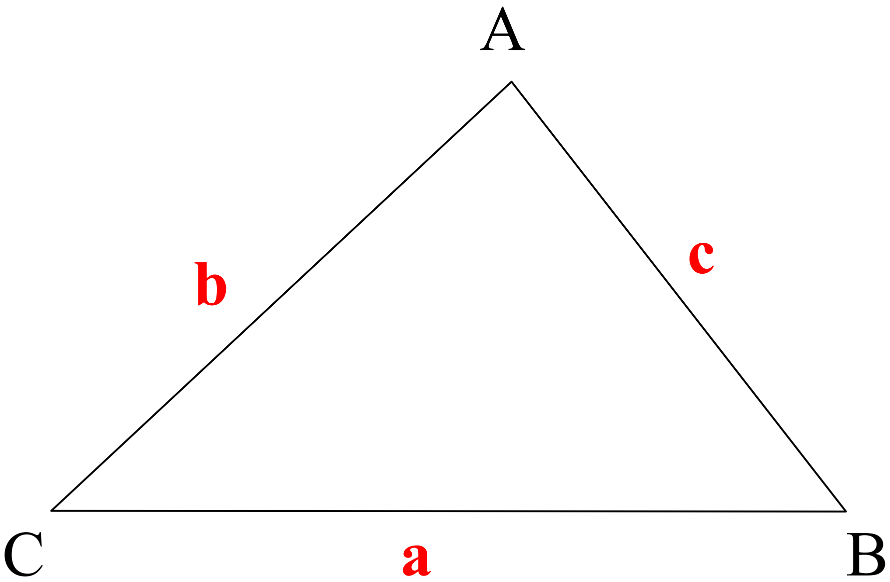
Objectives for Section 15-8

- to solve triangles for all missing sides and angles using both the law of cosines and the law of sines
- to find the area of a triangle using trigonometry

Law of Sines

In any triangle ABC,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

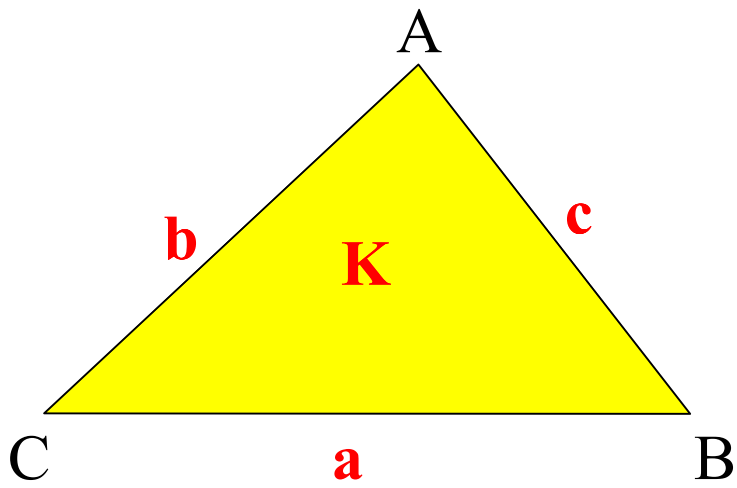


Solving General Triangles

- From geometry we know that triangle congruence can be proven with SSS, SAS, ASA, AAS but not with SSA.
- SSS, SAS, ASA, and AAS all give a single solution to a triangle.
- We use the following laws to solve each pattern:
  - SSS: Law of Cosines
  - SAS: Law of Cosines
  - ASA: Law of Sines
  - AAS: Law of Sines
- SSA is called the ambiguous case because it results in six possible solutions to the triangle: four if the given angle is acute and two if the given angle is right or obtuse.
- We use Law of Sines to solve the SSA pattern.

Areas of Triangles

The area K of a triangle ABC is given by:

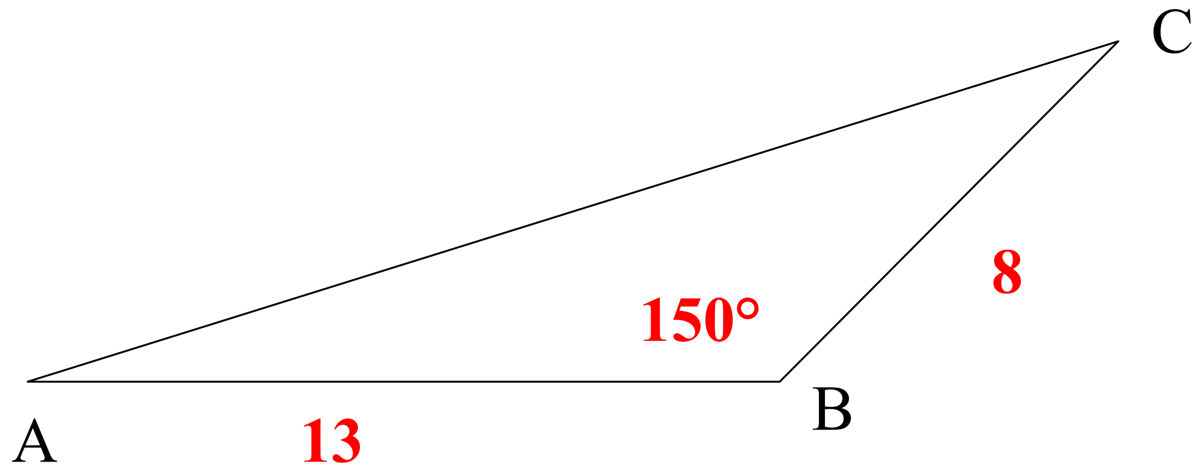


$$K = \frac{1}{2} bc \sin A \qquad K = \frac{1}{2} ac \sin B \qquad K = \frac{1}{2} ab \sin C$$

$$K = \frac{1}{2} a^2 \frac{\sin B \sin C}{\sin A} \qquad K = \frac{1}{2} b^2 \frac{\sin A \sin C}{\sin B} \qquad K = \frac{1}{2} c^2 \frac{\sin A \sin B}{\sin C}$$

$$K = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = \frac{1}{2}(a+b+c)$$

Examples for 1-10



SAS is solved first with Law of Cosines

$$b^2 = 8^2 + 13^2 - (2)(8)(13)(\cos 150)$$

b = 20.3

$$\frac{\sin 150}{20.3} = \frac{\sin C}{13} = \frac{\sin A}{8}$$

$$\sin C = \frac{(13)(0.5)}{20.3} = 0.3202$$

$$\sin A = \frac{(8)(0.5)}{20.3} = 0.1970$$

∠C = 18.7°, ∠A = 11.3°

Examples for 1-10

$a = 14, \angle A = 25°, \angle B = 75°, b = ?$

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin 25}{14} = \frac{\sin 75}{b}$$

$$\frac{0.4226}{14} = \frac{0.9659}{b}$$

$$b = \frac{(0.9659)(14)}{0.4226} = 32.0$$

Examples for 11-16

$$\angle A = 25°, \angle B = 50°, b = 30$$

$$K = \frac{1}{2} b^2 \frac{\sin A \sin C}{\sin B}$$

$$\angle C = 105°$$

$$K = \frac{1}{2}(30)^2 \frac{(\sin 25°)(\sin 105°)}{\sin 50°}$$

$$\mathbf{K = 240}$$