

# Section 16-1

## Inverse Functions: Principal Values

## Objectives for Section 16-1

- to find the values of all angles with a given trigonometric function value
- to find the general formula for all values of an angle that satisfy a trigonometric function
- to find the value of inverse trigonometric functions
- to find the value of the inverse trigonometric functions of a trigonometric function
- to find the inverse trigonometric values of angle sums, differences, double angles and half angles

# Inverse Trigonometric Functions

- inverse cosine (Arc cosine): Arccos or  $\text{Cos}^{-1}$  is the way to find an angle that has the given value of cosine.  
 $y = \text{Cos}^{-1} x$  if and only if  $\cos y = x$  and  $0 \leq y \leq \pi$
- inverse sine (Arc sine): Arcsin or  $\text{Sin}^{-1}$  is the way to find an angle with the given value of sine.  $y = \text{Sin}^{-1} x$  if and only if  $\sin y = x$  and  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
- inverse tangent (Arc tangent): Arctan or  $\text{Tan}^{-1}$  is the way to find an angle with the given value of tangent.  
 $y = \text{Tan}^{-1} x$  if and only if  $\tan y = x$  and  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
- inverse cotangent (Arc cotangent): Arccot or  $\text{Cot}^{-1}$  is the way to find an angle with the given value of cotangent.  
 $y = \text{Cot}^{-1} x$  if and only if  $\cot y = x$  and  $0 < y < \pi$

# Inverse Trigonometric Functions

- inverse secant (Arc secant:  $\text{Arcsec}$  or  $\text{Sec}^{-1}$  is the way to find an angle with the given value of secant. If  $|x| \geq 1$ , then  $\text{Sec}^{-1} x = \text{Cos}^{-1} \frac{1}{x}$  if and only if  $\sec y = x$  and  $0 < y < \pi$
- inverse cosecant (Arc cosecant:  $\text{Arccsc}$  or  $\text{Csc}^{-1}$  is the way to find an angle with the given value of cosecant. If  $|x| \geq 1$ , then  $\text{Csc}^{-1} x = \text{Sin}^{-1} \frac{1}{x}$  if and only if  $\csc y = x$   
$$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

## Example for 1-6

$$\cos \alpha = 1$$

$$\cos^{-1} 1 = \alpha$$

$$\alpha = 0^\circ$$

$$\alpha = 0^\circ + 360k$$

## Example for 7-12

$$\cos x = -\frac{1}{2}$$

$$\cos^{-1}\left(-\frac{1}{2}\right) = x$$

$$x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$x = \frac{2\pi}{3} + 2k\pi, \frac{4\pi}{3} + 2k\pi$$

## Example for 13-24

$$\text{Arc cos} \frac{\sqrt{3}}{2}$$

$$\cos^{-1} \frac{\sqrt{3}}{2} = x$$

$$x = \frac{\pi}{6}, 30^\circ$$

## Example for 25-33

$$\text{Sin}^{-1}(\cos 180^\circ)$$

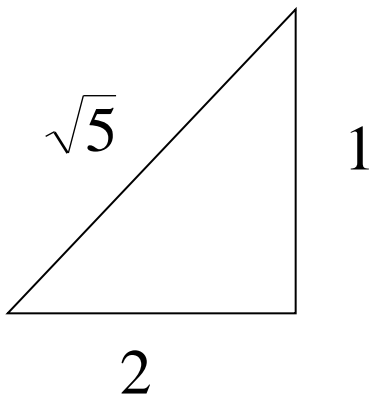
$$\text{Sin}^{-1}(-1)$$

$$-90^\circ$$

## Example for 34-41

$$\sin (2 \operatorname{Cot}^{-1} 2)$$

$$\sin \left( 2 \operatorname{Tan}^{-1} \frac{1}{2} \right)$$



$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$2 \left( \frac{1}{\sqrt{5}} \right) \left( \frac{2}{\sqrt{5}} \right)$$

$$\frac{4}{5}$$

## Example for 42-47

$$\sin \left[ \cos^{-1} \left( -\frac{1}{2} \right) - \sin^{-1} \frac{1}{2} \right]$$

$$\sin (120^\circ - 30^\circ)$$

$$\sin 90^\circ$$

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# Section 16-2

## Equations Involving Circular or Trigonometric Functions

## Objectives for Section 16-2

- to find the general solution and particular solutions to trigonometric equations

# Trigonometric Equations

- primary solutions: solutions in the interval  $0^\circ \leq x \leq 360^\circ$  or  $0 \leq x \leq 2\pi$
- general solutions: the formula for coterminal angles based on the period of the function you are solving

## Example for 1-38

$$\sin(\theta + 25^\circ) = 0$$

$$a = \theta + 25^\circ$$

$$\sin a = 0$$

$$a = 0^\circ \text{ or } 180^\circ$$

$$\theta + 25^\circ = 0^\circ \qquad \theta + 25^\circ = 180^\circ$$

$$\theta = -25^\circ \text{ or } 335^\circ \qquad \theta = 155^\circ$$

$$\theta = 155^\circ + 180k$$

## Example for 1-38

$$\cos 2\theta = 2 \sin^2 \theta$$

$$1 - 2 \sin^2 \theta = 2 \sin^2 \theta$$

$$1 = 4 \sin^2 \theta$$

$$\frac{1}{4} = \sin^2 \theta$$

$$\pm \frac{1}{2} = \sin \theta$$

**{30°, 150°, 210°, 330°}**

# Section 16-3

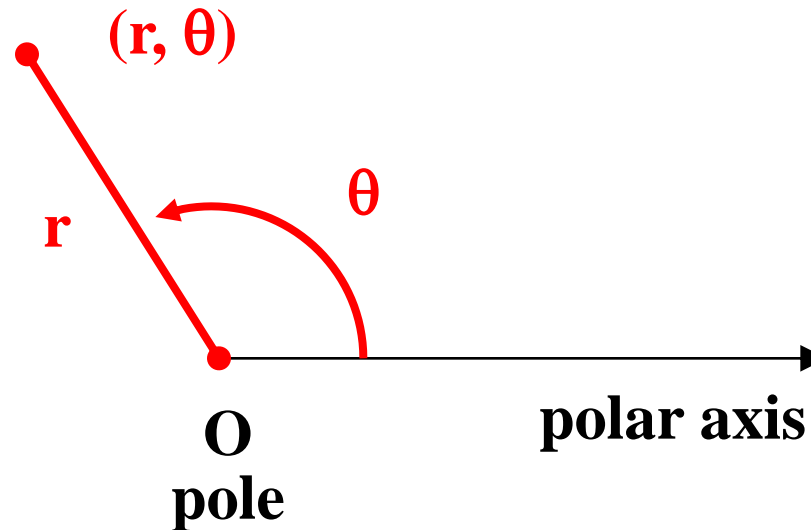
## Polar Coordinates: Polar Graphs

## Objectives for Section 16-3

- to convert from rectangular coordinates to polar coordinates
- to convert from polar coordinates to rectangular coordinates
- to write rectangular equations in polar form
- to write polar equations in rectangular form

# Polar Coordinates

- polar coordinate system: consists of a point  $O$  called the pole and a ray called the polar axis having  $O$  as its endpoint. The polar coordinates of any point are an ordered pair  $(r, \theta)$  where  $r = OP$  and  $\theta$  is the measure of the angle from the polar axis to  $OP$ . Negative values of  $r$  indicate that a point is on the ray opposite to the terminal side of  $\theta$ .



# Converting Coordinate Systems

- From Polar to Rectangular:

- $x = r \cos \theta$

- $y = r \sin \theta$

- From Rectangular to Polar:

- $r = \pm\sqrt{x^2 + y^2}$

- $\cos \theta = \frac{x}{r}, \quad \sin \theta = \frac{y}{r}$

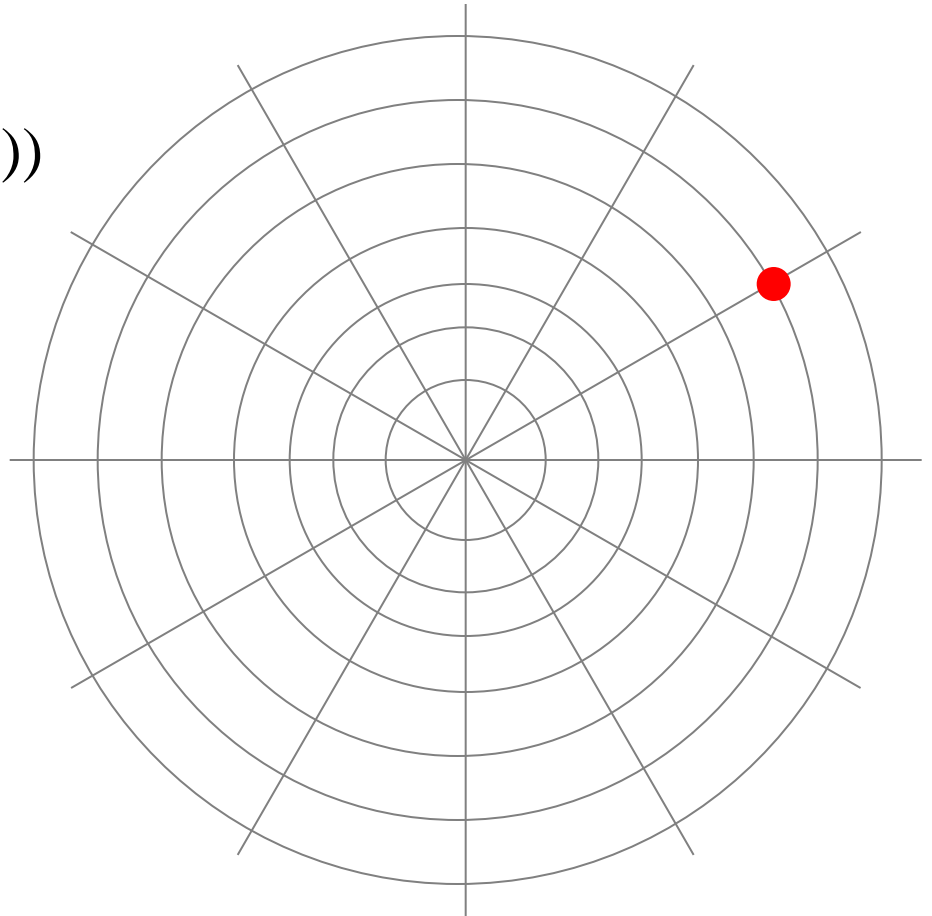
## Example for 1-8

$$(-6, -150^\circ)$$

$$(-6 \cos(-150^\circ), -6 \sin(-150^\circ))$$

$$\left(-6\left(-\frac{\sqrt{3}}{2}\right), -6\left(-\frac{1}{2}\right)\right)$$

$$(3\sqrt{3}, 3)$$



## Example for 9-17

$$(2, 120^\circ)$$

$$(2\cos 120^\circ, 2\sin 120^\circ)$$

$$\cos 120^\circ = -\frac{1}{2}, \quad \sin 120^\circ = \frac{\sqrt{3}}{2}$$

$$(-1, \sqrt{3})$$

## Example for 18-23

$$x^2 + y^2 = 6y$$

$$(r \cos \theta)^2 + (r \sin \theta)^2 = 6 r \sin \theta$$

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = 6 r \sin \theta$$

$$r^2 (\cos^2 \theta + \sin^2 \theta) = 6 r \sin \theta$$

$$r^2 = 6 r \sin \theta$$

$$**r = 6 \sin \theta**$$

## Example for 24-38

$$r(1 - \cos \theta) = 2$$

$$r\left(1 - \frac{x}{r}\right) = 2$$

$$r - x = 2$$

$$\sqrt{x^2 + y^2} - x = 2$$

$$\sqrt{x^2 + y^2} = x + 2$$

$$x^2 + y^2 = x^2 + 4x + 4$$

$$**y^2 = 4x + 4**$$

# Section 16-4

## Graphs of Complex Numbers

## Objectives for Section 16-4

- to convert rectangular complex numbers to polar form
- to convert polar complex numbers to rectangular form
- to find the product and quotient of polar complex numbers

# Geometry of Complex Numbers

- Complex plane: horizontal axis is the real axis, vertical axis is the imaginary axis
- absolute value of a complex number  $z = x + yi$  is  $\sqrt{x^2 + y^2}$   
the conjugate of  $z$  is  $\bar{z} = x - yi$  and  $-z = -x - yi$ .
- the polar form of a complex number  $z = x + yi$  is  $z = r (\cos \theta + i \sin \theta)$  where  $r$  and  $\theta$  have the same conversion formulas as they did for the rectangular coordinate plane.
- $\theta$  is called the amplitude or argument of  $z$
- $r = |z|$  is called the modulus of  $z$ .
- If  $w = a (\cos \alpha + i \sin \alpha)$  and  $z = b (\cos \beta + i \sin \beta)$ 
  - $wz = ab [\cos (\alpha + \beta) + i \sin (\alpha + \beta)]$
  - $\frac{w}{z} = \frac{a}{b} [\cos (\alpha - \beta) + i \sin (\alpha - \beta)]$

## Example for 1-8

$$1 - i\sqrt{3}$$

$$r = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1+3} = 2$$

$$\cos \theta = \frac{1}{2}, \quad \sin \theta = -\frac{\sqrt{3}}{2}$$

$$\theta = 300^\circ$$

$$\mathbf{2(\cos 300^\circ + i \sin 300^\circ)}$$

## Example for 9-16

$$3(\cos 30^\circ + i \sin 30^\circ)$$

$$3\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$\frac{3\sqrt{3}}{2} + \frac{3}{2}i$$

## Example for 17-22

$$w = 5(\cos 30^\circ + i \sin 30^\circ), z = 2(\cos 80^\circ + i \sin 80^\circ)$$

$$wz = (5)(2)[\cos(30^\circ + 80^\circ) + i \sin(30^\circ + 80^\circ)]$$

$$\mathbf{wz = 10[\cos 110^\circ + i \sin 110^\circ]}$$

$$\frac{w}{z} = \frac{5}{2}[\cos(30^\circ - 80^\circ) + i \sin(30^\circ - 80^\circ)]$$

$$\frac{w}{z} = \frac{5}{2}(\cos 310^\circ + i \sin 310^\circ)$$

# Section 16-5

## De Moivre's Theorem

## Objectives for Section 16-5

- to use De Moivre's Theorem to find the powers of complex numbers
- to use De Moivre's Theorem to find the roots of unity
- to use De Moivre's Theorem to find the roots of complex numbers

## De Moivre's Theorem

- If  $z = r (\cos \theta + i \sin \theta)$  and  $n$  is a positive integer then  $z^n = r^n (\cos n\theta + i \sin n\theta)$
- The roots of a complex number are found by working DeMoivre's Theorem in reverse

$$\sqrt[n]{r} \left( \cos \frac{\theta + k \bullet 360^\circ}{n} + i \sin \frac{\theta + k \bullet 360^\circ}{n} \right) \quad (k = 0, 1, 2, \dots, n - 1)$$

- The  $n$ th roots of the number 1 are called the  $n$ th roots of unity and can be found with the formula:

$$\cos \frac{k \bullet 360^\circ}{n} + i \sin \frac{k \bullet 360^\circ}{n} \quad (k = 0, 1, 2, \dots, n - 1)$$

## Example for 1-14

$$(1 - i\sqrt{3})^7$$

$$r = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1+3} = 2$$

$$\cos \theta = \frac{1}{2} \quad \sin \theta = -\frac{\sqrt{3}}{2}$$

$$\theta = 300^\circ$$

$$2^7 (\cos 7 \cdot 300^\circ + i \sin 7 \cdot 300^\circ)$$

$$128 \left( \frac{1}{2} - \frac{\sqrt{3}}{2} i \right)$$

$$64 - 64i\sqrt{3}$$

## Example for 15-19

The cube roots of  $4\sqrt{3} - 4i$

$$r = \sqrt{(4\sqrt{3})^2 + (-4)^2} = \sqrt{48 + 16} = 8$$

$$\cos \theta = \frac{4\sqrt{3}}{8} \quad \sin \theta = -\frac{4}{8}$$

$$\theta = 330^\circ$$

$$r^3 (\cos 3\theta + i \sin 3\theta) = 8 (\cos 330 + i \sin 330)$$

$$3\theta = 330^\circ + k \cdot 360^\circ \text{ (k is an integer)}$$

$$\theta = 110^\circ + k \cdot 120^\circ \text{ (k is an integer)}$$

$$\theta = 110^\circ, 230^\circ, 350^\circ$$

$$r^3 = 8$$

$$r = 2$$

$$\mathbf{2(\cos 110^\circ + i \sin 110^\circ)}$$

$$\mathbf{2(\cos 230^\circ + i \sin 230^\circ)}$$

$$\mathbf{2(\cos 350^\circ + i \sin 350^\circ)}$$

## Example for 15-19

The cube roots of unity.

$$\cos \frac{0 \bullet 360^\circ}{3} + i \sin \frac{0 \bullet 360^\circ}{3} = \cos 0^\circ + i \sin 0^\circ$$

$$\cos \frac{1 \bullet 360^\circ}{3} + i \sin \frac{1 \bullet 360^\circ}{3} = \cos 120^\circ + i \sin 120^\circ$$

$$\cos \frac{2 \bullet 360^\circ}{3} + i \sin \frac{2 \bullet 360^\circ}{3} = \cos 240^\circ + i \sin 240^\circ$$

$$\mathbf{1 + 0i = 1}$$

$$\mathbf{-\frac{1}{2} + \frac{\sqrt{3}}{2}i}$$

$$\mathbf{-\frac{1}{2} - \frac{\sqrt{3}}{2}i}$$

# Section 16-6

## Vectors in the Plane

## Objectives for Section 16-6

- to draw vectors, vector sums and to find the magnitude of the resultant
- to find the measure of the angle between two vectors
- to find an orthogonal unit vector

## Vectors in the Plane

- $\mathbf{i}$  is the horizontal unit vector
- $\mathbf{j}$  is the vertical unit vector
- Every vector  $\mathbf{u}$  can be expressed as the resultant of the unit vectors. If  $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$  the number  $a$  represents the x-component of the vector and the number  $b$  represents the y-component of the vector. This is called component form.
- If  $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ ,  $\mathbf{v} = c\mathbf{i} + d\mathbf{j}$  and  $t$  is a scalar, then
  - $\mathbf{u} = \mathbf{v}$  if and only if  $a = c$  and  $b = d$
  - $\mathbf{u} + \mathbf{v} = (a + c)\mathbf{i} + (b + d)\mathbf{j}$
  - $t\mathbf{u} = ta\mathbf{i} + tb\mathbf{j}$
  - $\|\mathbf{u}\| = \sqrt{a^2 + b^2}$
- The dot product of two nonzero vectors  $\mathbf{u}$  and  $\mathbf{v}$  is defined to be:  $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$  where  $\theta$  is the angle between

## Vectors in the Plane

- If  $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$  and  $\mathbf{v} = c\mathbf{i} + d\mathbf{j}$  then  $\mathbf{u} \cdot \mathbf{v} = ac + bd$
- Vectors are orthogonal (either vector is zero or they are perpendicular) if and only if  $\mathbf{u} \cdot \mathbf{v} = 0$ .

## Example for 1-8

$$\mathbf{u} = 4\mathbf{i} + 3\mathbf{j}, \mathbf{v} = 2\mathbf{i} - \mathbf{j}; \mathbf{w} = \mathbf{u} + \mathbf{v}$$

$$\mathbf{w} = (4 + 2)\mathbf{i} + (3 - 1)\mathbf{j}$$

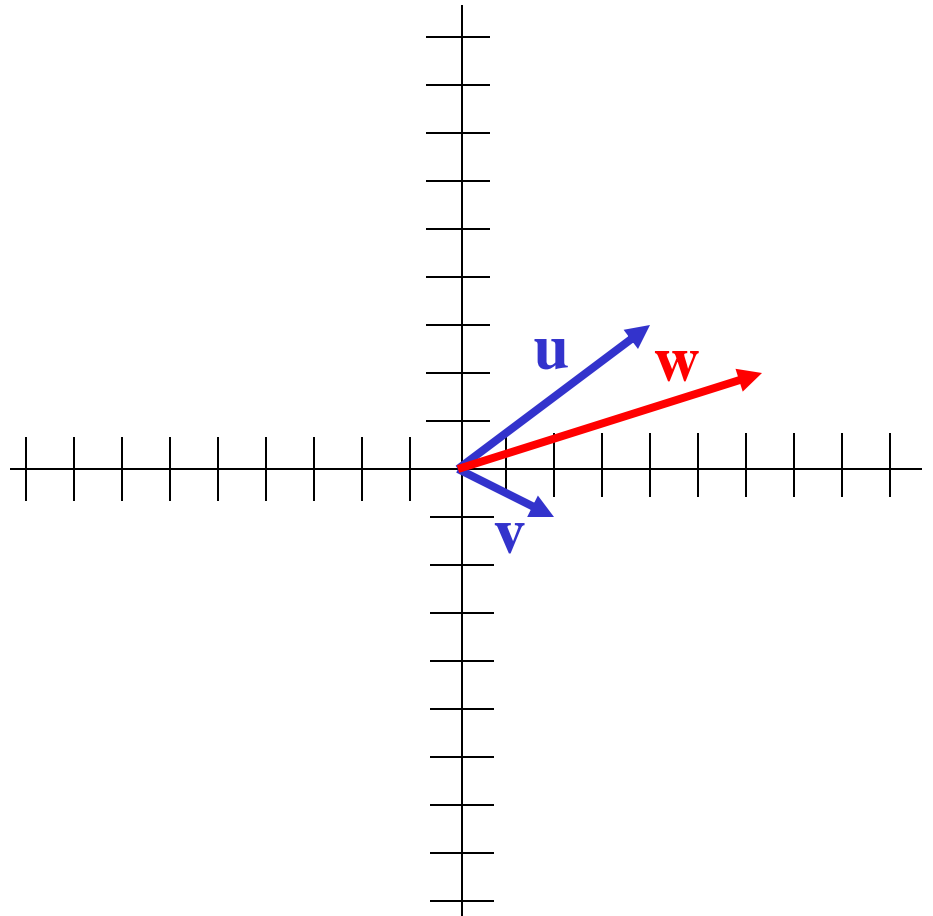
$$\mathbf{w} = 6\mathbf{i} + 2\mathbf{j}$$

$$\|\mathbf{w}\| = \sqrt{6^2 + 2^2}$$

$$\|\mathbf{w}\| = 2\sqrt{10}$$

$$\cos \gamma = \frac{6}{2\sqrt{10}}$$

$$\gamma = 18.4^\circ$$



## Example for 1-8

$$\|\mathbf{u}\| = 10, 100^\circ; \|\mathbf{v}\| = 6, 40^\circ$$

$$\mathbf{w} = (10\cos 100^\circ + 6\cos 40^\circ)\mathbf{i} + (10\sin 100^\circ + 4\sin 40^\circ)\mathbf{j}$$

$$\|\mathbf{w}\| = \sqrt{6^2 + 10^2 - 2(6)(10)\cos 120^\circ}$$

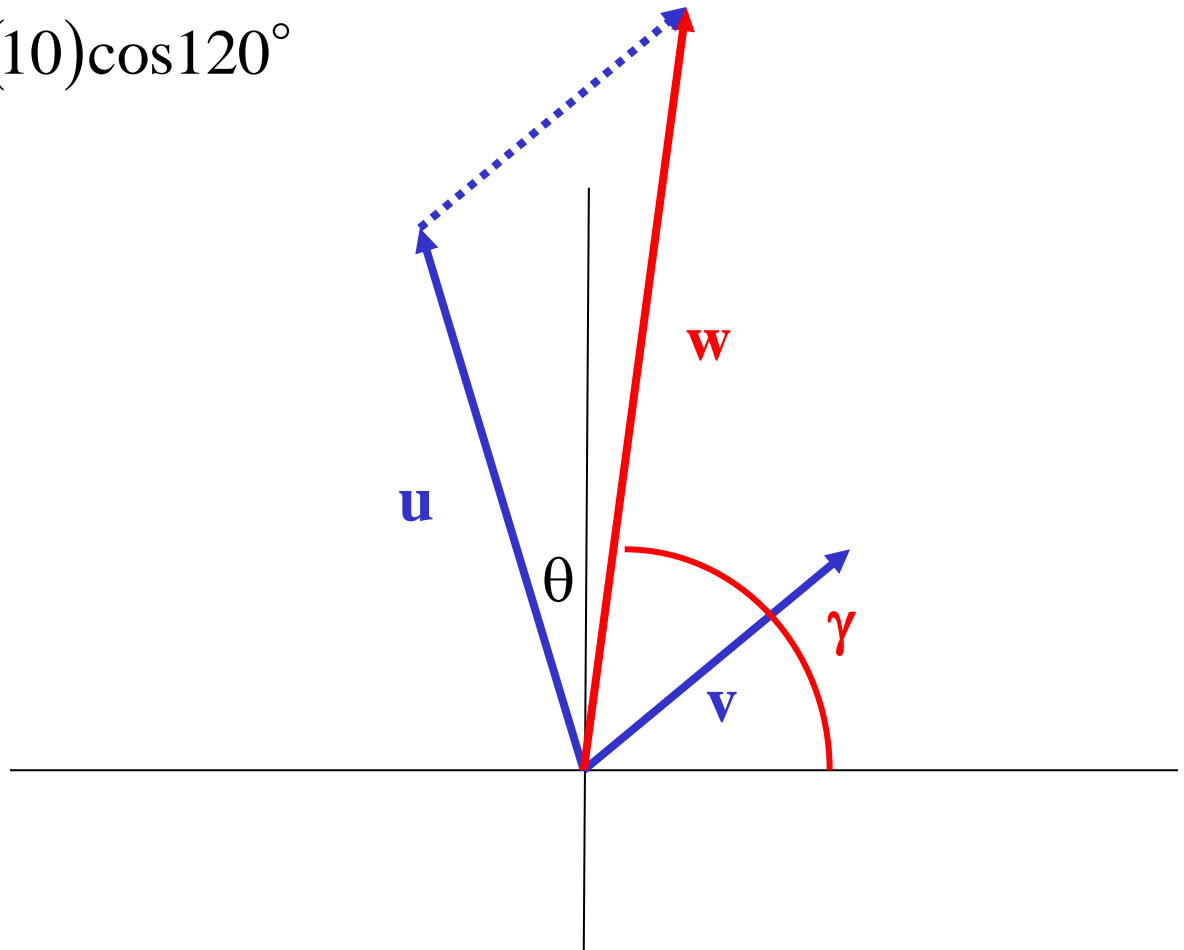
$$\|\mathbf{w}\| = 14$$

$$\gamma = 100^\circ - \theta$$

$$\frac{\sin \theta}{6} = \frac{\sin 120^\circ}{\|\mathbf{w}\|}$$

$$\theta = \sin^{-1}\left(\frac{6\sin 120^\circ}{\|\mathbf{w}\|}\right)$$

$$\theta \approx 21.8^\circ \quad \gamma = 72.2$$



## Example for 9-12

$$\mathbf{u} = 4\mathbf{i} + 3\mathbf{j}, \mathbf{v} = 2\mathbf{i} - \mathbf{j}$$

$$\mathbf{u} \cdot \mathbf{v} = (4)(2) + (3)(-1)$$

$$\mathbf{u} \cdot \mathbf{v} = 8 - 3 = 5$$

$$\|\mathbf{u}\| = \sqrt{4^2 + 3^2} = 5$$

$$\|\mathbf{v}\| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$$

$$5 = (5)(\sqrt{5})\cos\gamma$$

$$\frac{1}{\sqrt{5}} = \cos\gamma$$

$$\gamma = 63.4^\circ$$

## Example for 13-16

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$$

$$\mathbf{u} \cdot \mathbf{v} = 3a - 4b = 0$$

$$a = 4, b = 3$$

$$\mathbf{u} = 4\mathbf{i} + 3\mathbf{j}$$

$$\|\mathbf{u}\| = \sqrt{4^2 + 3^2} = 5$$

$$\text{unit vector} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$$

## Example for 17-24

$$\|\mathbf{u}\| = 8, \alpha = 20^\circ; \|\mathbf{v}\| = 17, \beta = 80^\circ$$

$$\mathbf{u} = (8 \cos 20)\mathbf{i} + (8 \sin 20)\mathbf{j}$$

$$\mathbf{v} = (17 \cos 80)\mathbf{i} + (17 \sin 80)\mathbf{j}$$

$$\mathbf{u} = 7.5\mathbf{i} + 2.7\mathbf{j}, \mathbf{v} = 3.0\mathbf{i} + 16.7\mathbf{j}$$

$$\mathbf{u} + \mathbf{v} = 10.5\mathbf{i} + 19.4\mathbf{j}$$

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{(10.5)^2 + (19.4)^2} = 22.1$$

$$\cos \theta = \frac{10.5}{22.1}, \sin \theta = \frac{19.4}{22.1}$$

$$\theta = 62^\circ$$

# Section 16-7

## Applications of Vectors

## Objectives for Section 16-7

- to find bearing and magnitude of resultant vectors

# Vectors

- vector quantity: any quantity with both magnitude and direction
- $\vec{AB}$  : read as the vector  $AB$  has an initial point at  $A$  and a terminal point at  $B$ . Boldface letters such as **u** and **v** also denote vectors
- equivalent vectors have the same magnitude and direction
- zero vector is denoted by **0**

## Vector Addition

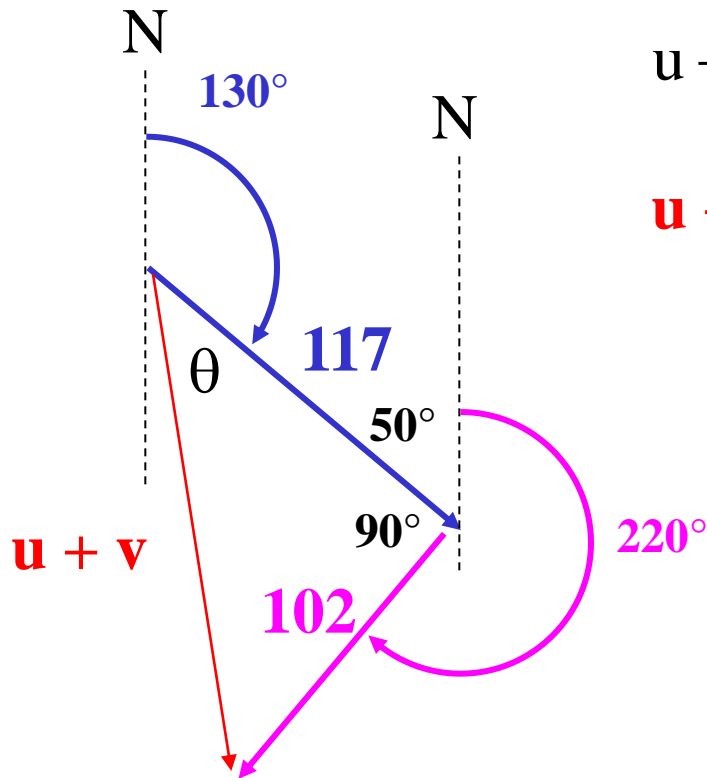
- Given two vectors  $\mathbf{u}$  and  $\mathbf{v}$ , you can find their sum or resultant, by using either the triangle method or the parallelogram method.
- Triangle Method: place the initial point of  $\mathbf{v}$  at the terminal point of  $\mathbf{u}$ . Then  $\mathbf{u} + \mathbf{v}$  is the vector extending from the initial point of  $\mathbf{u}$  to the terminal point of  $\mathbf{v}$ .
- Parallelogram Method: form a parallelogram with  $\mathbf{u}$  and  $\mathbf{v}$  as adjacent sides starting from a common point. Then  $\mathbf{u} + \mathbf{v}$  extends from that common point to the opposite vertex of the parallelogram.

## Scalar Multiplication, Magnitude & Bearing

- To multiply the vector  $\mathbf{v}$  by the real number  $t$ , multiply the length of  $\mathbf{v}$  by  $|t|$  and reverse the direction if  $t < 0$ .
- $\|\mathbf{v}\|$ : is the magnitude of  $\mathbf{v}$ . You can use law of cosines to find magnitude.
- The bearing of a vector  $\mathbf{v}$  is the angle measured clockwise from due north around to  $\mathbf{v}$ . You can use law of sines to find bearing.

## Example for 1-13

**u** has magnitude 117 and bearing  $130^\circ$ , **v** has magnitude 102 and bearing  $220^\circ$ , find the magnitude of  $\mathbf{w} = \mathbf{u} + \mathbf{v}$  to 3 significant digits and the bearing of **w** to the nearest tenth of a degree.



$$u + v = \sqrt{117^2 + 102^2 - 2(117)(102)(\cos 90^\circ)}$$

$$u + v = 155$$

$$\frac{\sin \theta}{102} = \frac{\sin 90^\circ}{155}$$

$$\sin \theta = 0.6581$$

$$\theta = 41.2^\circ$$

$$\text{bearing } u + v = 171.2^\circ$$