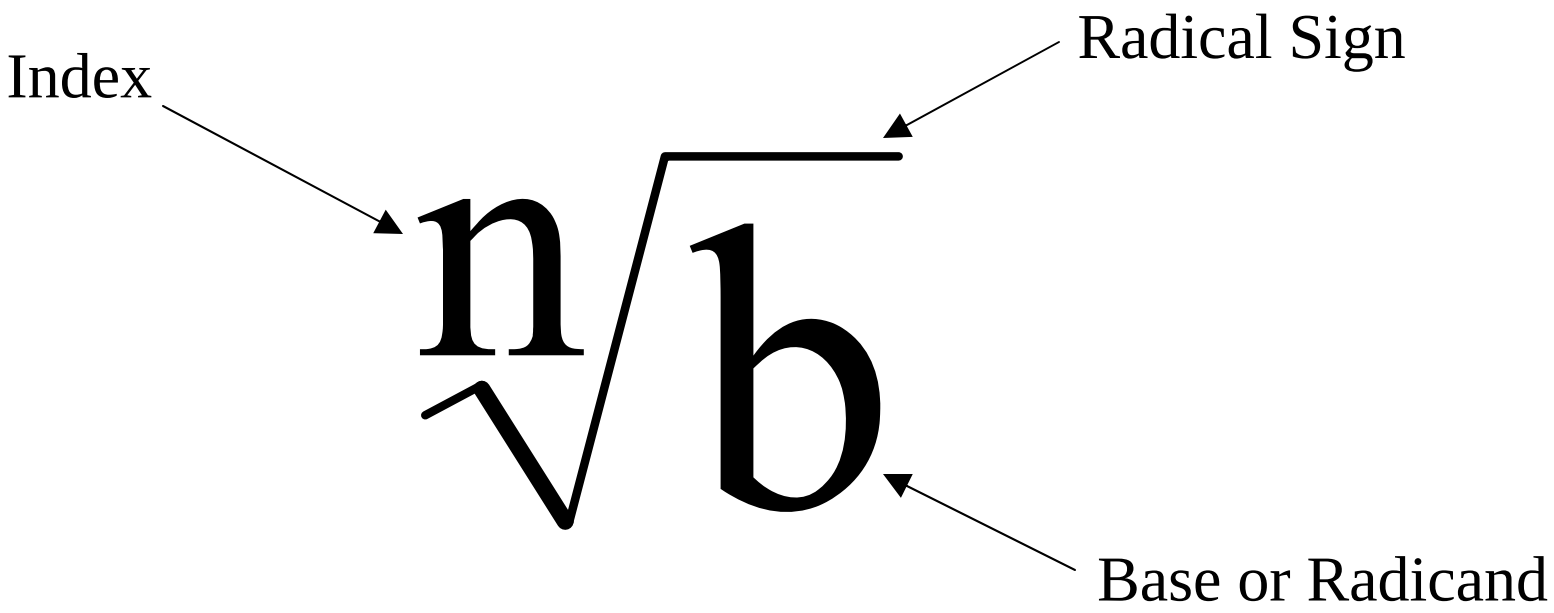
Section 6-1

Roots and Real Numbers

Objectives

- simplify radical expressions and identify all real roots
- find the real roots of an equation
- find the values of a variable that make an open sentence true
- find the values that make a radical have real roots

Radical Symbols



Roots

Index is even:

- base (radicand) > 0, then there are two real roots. The principal (positive) real root and the secondary (negative) real root.
  - When simplifying always use the principal root.
  - When solving you must give both roots.
- base (radicand) = 0, then there is one real root, 0.
- base (radicand) < 0, then there are no real roots.

Index is odd:

- base (radicand) > 0, then there is one real positive root.
- base (radicand) = 0, then there is one real root = 0.
- base (radicand) < 0, then there is one real negative root

Properties of Radicals

$$\left(\sqrt[n]{b}\right)^n = b$$

$$\sqrt[n]{b^n} = b, \text{ if } n \text{ is odd.}$$

$$\sqrt[n]{b^n} = |b|, \text{ if } n \text{ is even.}$$

Problems 1-14: Simplify each expression. If the expression does not represent a real number say so.

I. Identify whether the index is even or odd.

II. Identify whether the base (radicand) is positive or negative

III. Determine the number and type of roots based on the table given earlier on roots and on the properties of radicals.

IV. Find the correct value or variable root.

Example Problem: 3a. $\sqrt{81}$

I. Index is even.

II. Base (radicand) is positive

III. Two real roots. When simplifying use the principal (positive) root.

IV. 9

Example Problem: 3b. $-\sqrt{81}$

I. Index is even.

II. Base (radicand) is positive

III. Two real roots. When simplifying use the principal (positive) root.

IV. $-(9) = -9$

Example Problem: 3c. $\sqrt{-81}$

I. Index is even.

II. Base (radicand) is negative

III. no real roots

Example Problem: 3d. $\sqrt[4]{81}$

I. Index is even.

II. Base (radicand) is positive

III. Two real roots. When simplifying use the principal (positive) root.

IV. 3

Example Problem: 13a. $\sqrt{a^2}$

I. Index is even

II. We do not know the sign of a.

III. Apply the third property of radicals

IV. $|a|$

Example Problem: 13b. $\sqrt{a^4}$

I. Index is even

II. We do not know the sign of a.

III. Apply the third property of radicals

IV. a^2

Example Problem: 13c. $\sqrt[3]{a^6}$

I. Index is odd

II. We do not know the sign of a.

III. Apply the second property of radicals

IV. a^2

Example Problem: 13d. $\sqrt[6]{a^6}$

I. Index is even

II. We do not know the sign of a.

III. Apply the third property of radicals

IV. $|a|$

Problems 15-26: Find the real roots of each equation. If there are none, say so.

I. Isolate the variable on one side.

II. Take the square root of both sides.

III. Identify the sign of the base (radicand).

IV. Provide the appropriate principal and secondary roots since you are solving.

Example Problem: 19. $9x^2 = 4$

I. $x^2 = \frac{4}{9}$

II. $\sqrt{x^2} = \sqrt{\frac{4}{9}}$

III. Base is positive so there are two real roots.

IV. $x = \pm\frac{2}{3}$

Example Problem; 17. $x^2 + 9 = 0$

I. $x^2 = -9$

II. $\sqrt{x^2} = \sqrt{-9}$

III. The base is negative.

IV. no real roots

Problems 27-32: For what values of the variable is each equation true.

I. Determine whether the index is even or odd.

II. If the index is even, then find the values of the variable that make the root positive.

III. If the index is odd then all real numbers will work.

Example Problem: 27. $\sqrt{(x+5)^2} = x+5$

I. The index is even

II. Solve $x + 5 \geq 0$

$x \geq -5$

Problems 33-34: For what values of x does each expression represent a real number.

I. Set the base (radicand) ≥ 0 and solve.

II. Some of the problems represent simple inequalities and will be solved like problems in Ch 2

III. Some of the problems represent quadratic inequalities and will be solved like problems in Ch 4

Example Problem: 33d. $\sqrt{x^2 - 1}$

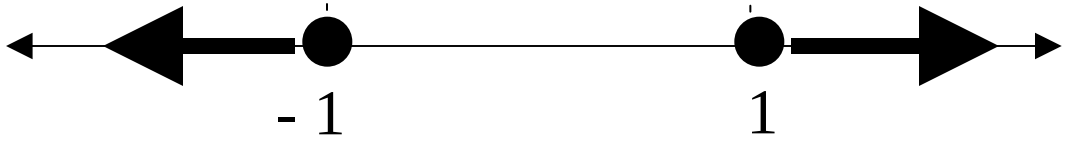
I. $x^2 - 1 \geq 0$

II. $(x + 1)(x - 1) \geq 0$

III. $x + 1$

$x - 1$

Answer



Section 6-2

Properties of Radicals

Objectives

- simplify and rationalize products and quotients of radicals
- give decimal approximations of radicals
- evaluate radical expressions

More Properties of Radicals

A radical is simplified if :

1. no radicand contains a factor other than 1 that has a power equal to the index
2. and every denominator has been rationalized so that no radicand is a fraction and no radical is in the denominator.

$$\sqrt[n]{ab} = \sqrt[n]{a} \bullet \sqrt[n]{b}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

$$\sqrt[nq]{b} = \sqrt[n]{\sqrt[q]{b}} \text{ or } \sqrt[q]{\sqrt[n]{b}}$$

$$\sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$$

Simplifying Radicals Review

Prime factor the radicand. Any factor which has an exponent equal to the index may be placed as a factor in front of the radical. All other factors must remain under the radical.

$$\sqrt[3]{1500} = \sqrt[3]{\left(5^3\right)\left(2^2\right)(3)} = 5\sqrt[3]{12}$$

Rationalizing Review

You must multiply the fraction by a form of the number one so that the radicand of the denominator has a power equal to its index.

$$\frac{\sqrt[4]{x^3}}{\sqrt[4]{y}}\left(\frac{\sqrt[4]{y^3}}{\sqrt[4]{y^3}}\right)=\frac{\sqrt[4]{x^3y^3}}{\sqrt[4]{y^4}}=\frac{\sqrt[4]{x^3y^3}}{|y|}$$

Problems 1- 32, 39 - 46 & 59 - 66: Simplify

I. Simplify any radicals using the properties of radicals, remember the rules for roots of even indices especially when working with variables.

II. Cancel any common factors

III. Rationalize any denominators

Example Problem: 29. $\dfrac{\sqrt[3]{60}}{\sqrt[3]{36}}$

I. $\dfrac{\left(\sqrt[3]{12}\right)\left(\sqrt[3]{5}\right)}{\left(\sqrt[3]{12}\right)\left(\sqrt[3]{3}\right)}$

II. $\dfrac{\left(\sqrt[3]{5}\right)\left(\sqrt[3]{3^2}\right)}{\left(\sqrt[3]{3}\right)\left(\sqrt[3]{3^2}\right)}$

III. $\dfrac{\sqrt[3]{45}}{3}$

Problems 33- 38: Give a decimal approximation to the nearest hundredth for each radical.

I. Locate the perfect powers on either side of the radicand.

II. Estimate how far between the numbers your radicand is.

III. You may check it with a calculator.

Example Problem: 33. $\sqrt{39}$

I. The radicand 39 is between the perfect squares 36 and 49.

II. 42 would be a little more than halfway between 36 and 49 so it could be expected to have a square root close to 6.5.

III. Since 39 is a little less than halfway between 36 and 42 it could be expected to have a square root of 6.24.

Problems 51-58: Evaluate the following radicals if $x = 4$, $y = 3$ & $z = 8$.

I. Substitute the values in for the appropriate variables

II. Follow order of operations and the properties of radicals to simplify.

III. Remember to rationalize any answers

Example Problem: 53. $\sqrt[3]{x^{-1}+z^{-1}}$

I. $x = 4$ & $z = 8$ $\sqrt[3]{4^{-1}+8^{-1}}$

II. $\sqrt[3]{\frac{1}{4}+\frac{1}{8}} = \sqrt[3]{\frac{2}{8}+\frac{1}{8}} = \sqrt[3]{\frac{3}{8}}$

III. $\frac{\sqrt[3]{3}}{2}$

Section 6-3

Sums of Radicals

Objectives

- to perform basic radical arithmetic with both numerical and variable radicals

Radical Addition

- Radicals are considered similar and can be combined through addition and subtraction only if they have identical indices and identical radicands.
- If these two criteria are met, then the radicals can be combined and the number of similar radicals is expressed as a coefficient in front of the radical.

Problems 1-42: Simplify.

I. Simplify and rationalize each term in the statement.

II. Identify similar terms and combine.

Example Problem: 39. $\sqrt{10a} - \frac{\sqrt{5a}}{\sqrt{2}} + \sqrt{\frac{2a}{5}}$

I. $\sqrt{10a} - \left(\frac{\sqrt{5a}}{\sqrt{2}}\right)\left(\frac{\sqrt{2}}{\sqrt{2}}\right) + \left(\frac{\sqrt{2a}}{\sqrt{5}}\right)\left(\frac{\sqrt{5}}{\sqrt{5}}\right)$

$\sqrt{10a} - \frac{\sqrt{10a}}{2} + \frac{\sqrt{10a}}{5}$

II. $\frac{10\sqrt{10a}}{10} - \frac{5\sqrt{10a}}{10} + \frac{2\sqrt{10a}}{10}$

$\frac{7\sqrt{10a}}{10}$

Section 6-4

Binomials Containing Radicals

Objectives

- to simplify and rationalize products of binomials involving radicals

Conjugates

- To rationalize a fraction with a binomial radical in the denominator we will take advantage of the factoring pattern for $(a + b)(a - b)$.
- Because this pattern has inside and outside products that cancel we will use factors that follow this pattern to eliminate the radicals.
- Terms like $(a + b)$ & $(a - b)$ are called conjugates.

Problems1-32: Simplify

I. If the problem is a product of binomials then FOIL as you would any other product of binomials.

II. If the problem is a quotient and there is a binomial radical in the denominator, then multiply both the numerator and the denominator by the conjugate.

Example Problems: 5. $\left(1+\sqrt{2}\right)\left(3+\sqrt{2}\right)$

31. $\dfrac{\sqrt{5}+1}{\sqrt{5}-3}$

I. FOIL $(1)(3)+(1)\left(\sqrt{2}\right)+\left(\sqrt{2}\right)(3)+\left(\sqrt{2}\right)\left(\sqrt{2}\right)$

$3+\sqrt{2}+3\sqrt{2}+2$

$5+4\sqrt{2}$

II. multiply by the conjugate of the denominator

$$\left(\frac{\sqrt{5}+1}{\sqrt{5}-3}\right)\left(\frac{\sqrt{5}+3}{\sqrt{5}+3}\right)=\frac{5+3\sqrt{5}+\sqrt{5}+3}{5+3\sqrt{5}-3\sqrt{5}-9}=\frac{8+4\sqrt{5}}{-4}$$

$-2-\sqrt{5}$

Section 6-5

Equations Containing Radicals

Objectives

- to solve equations involving radical expressions

Solving with Radicals

I. Isolate the radical.

II. Raise both sides of the equation to the power of the index.

III. Solve the resulting equation by the appropriate method.

IV. If the equation has two radical terms, then perform steps I & II twice before solving the resulting equation.

V. Check answers in the original problem to identify any extraneous roots.

Example Problem: 27. $\sqrt{2n-5}-\sqrt{3n+4}=2$

I. isolate the first radical $\sqrt{2n-5}=2+\sqrt{3n+4}$

II. square both sides and simplify

$$\left(\sqrt{2n-5}\right)^2=\left(2+\sqrt{3n+4}\right)^2$$

$$2n-5=4+4\sqrt{3n+4}+3n+4$$

$$2n-5=3n+8+4\sqrt{3n+4}$$

I. isolate the second radical $-n-13=4\sqrt{3n+4}$

II. square both sides and simplify

$$(-n-13)^2=\left(4\sqrt{3n+4}\right)^2$$

$n^2 + 26n + 169 = 16(3n + 4)$

$n^2 + 26n + 169 = 48n + 64$

$n^2 - 22n + 105 = 0$

$(n - 7)(n - 15) = 0$

n = 7 or 15 but neither answer works in the original problem so the final answer is ϕ

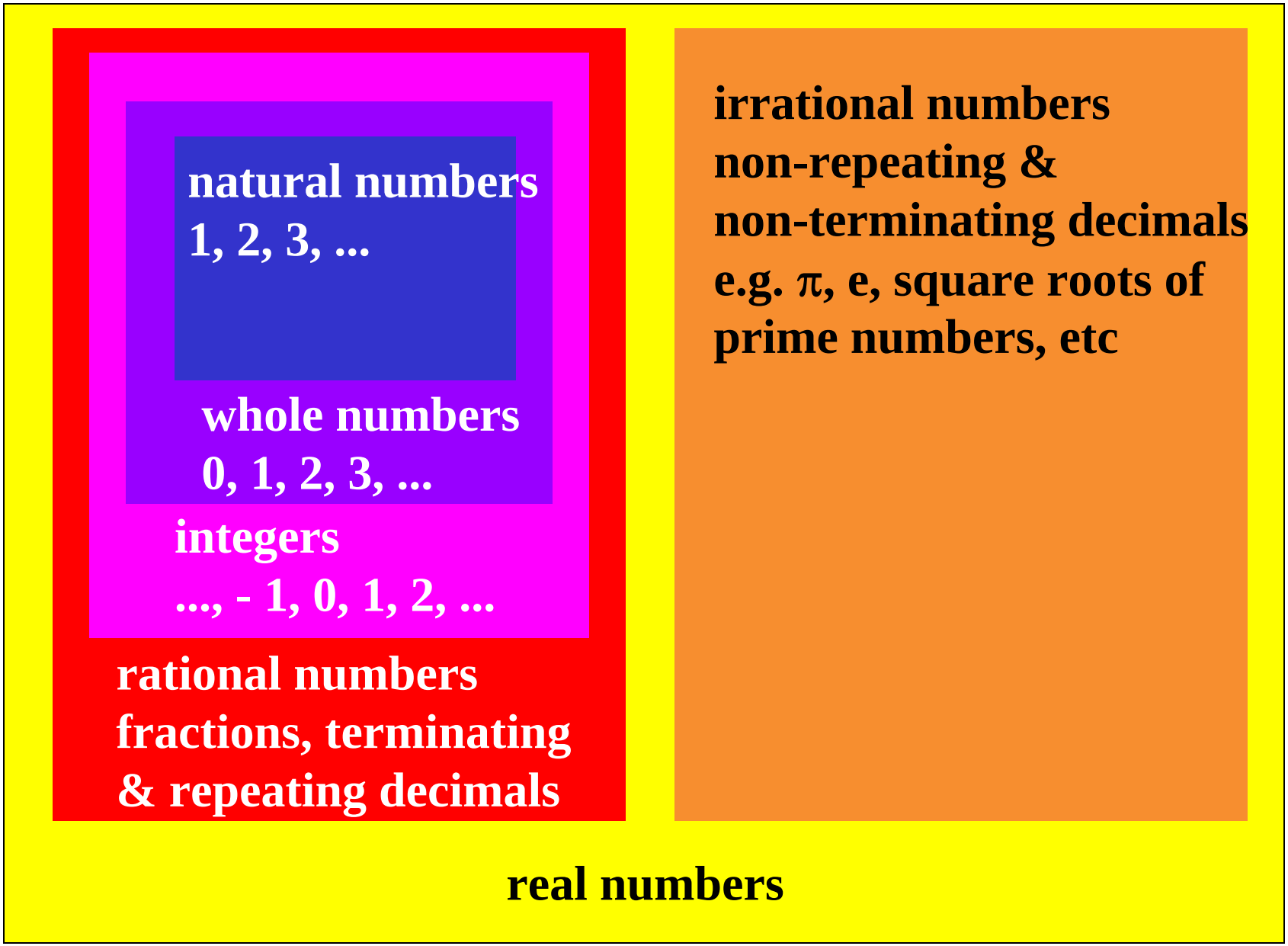
Section 6-6

Rational and Irrational Numbers

Objectives

- to classify expressions as either rational or irrational
- to write fractions as repeating or terminating decimals
- to write decimals as reduced fractions
- to identify both a rational and an irrational number between a given pair of values

The Set of Real Numbers



Choosing between rational and irrational

- If any term in a simplified sum is irrational, then the sum is irrational.
- If any factor in a simplified product is irrational, then the product is irrational.

Problems 1-6: Classify each expression as rational or irrational.

I. Simplify the sums and products according to all of the laws and properties you have learned and then use the previous two statements to classify the result

Problems 7-10: Write each fraction as a decimal.

I. Divide to find either the terminating or repeating decimal that represents each fraction.

II. Be careful if you use a calculator to make sure that it has not rounded the display.

Problems 11-22: Write each decimal as a fraction in lowest terms.

I. If the decimal terminates, then write the number without a decimal over the power of ten represented by the position of the decimal

II. If the decimal repeats, then set up a system of equations to solve for the fraction.

III. Let x be the unknown fraction and set it equal to the decimal.

IV. Multiply both sides of the equation by a power of ten that will move one whole pattern to the left of the decimal.

V. Subtract, solve and reduce.

Example Problems: 11. 5.06 & 17. $0.8\overline{3}$

I. $5.06 = \frac{506}{100} = \frac{253}{50}$

II. only the 3 repeats

III. $x = 0.8\overline{3}$

IV. $100x = 83.\overline{3}$

$10x = 8.\overline{3}$

$90x = 75$

$x = \frac{75}{90} = \frac{5}{6}$

Problems 23-31: Find (a) a rational & (b) an irrational number between the two given values.

I. Use part b and part c of question 5 as a guide for writing decimal numbers that are rational and irrational.

Example Problem: 23. 0.1 & 0.2

I. (a) 0.15

(b) 0.12345678910111213141 5...

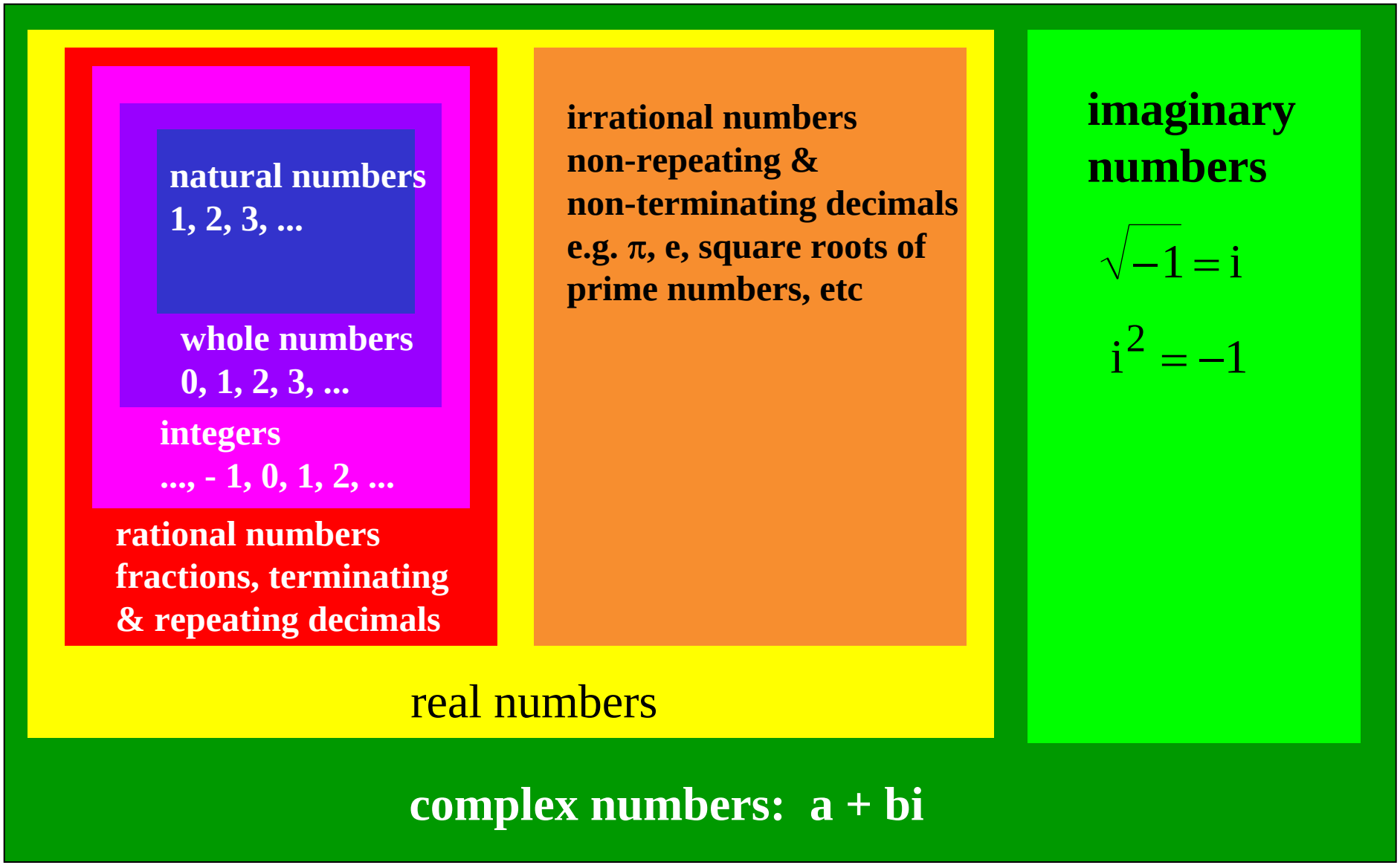
Section 6-7

The Imaginary Number ***i***

Objectives

- to simplify imaginary numbers
- to solve simple equations for imaginary solutions

The Set of Complex Numbers



Properties of iota (yo͞ - tă)

- $i^1 = i$
- $i^2 = -1$
- $i^3 = -i$
- $i^4 = 1$
- the pattern repeats itself through these four values for all the powers of iota.

- We know have a pattern for factoring sums of perfect squares $a^2 + b^2 = (a + bi)(a - bi)$

Problems 1-30, 37-54: Simplify.

I. Simplify each radical term individually.

II. Following order of operations simplify and combine similar terms.

III. Express answers in simplest terms.

Example: #51 $\sqrt{-x^5} + x\sqrt{-25x^2} - x^2\sqrt{-25x}$

I. $x^2 i\sqrt{x} + 5x^2 i - 5x^2 i\sqrt{x}$

II. $5x^2 i + \left(x^2 i\sqrt{x} - 5x^2 i\sqrt{x}\right)$

III. $5x^2 i - 4x^2 i\sqrt{x}$

Problems 31-36: Solve

I. Move all terms to one side.

II. Factor out a GCF

III. Follow the pattern for sum of perfect squares
$a^2 + b^2 = (a + bi)(a - bi)$

IV. Set each factor equal to zero and solve for the variable.

Example #33. $2w^2 = -98$

I. $2w^2 + 98 = 0$

II. $2(w^2 + 49) = 0$

III. $2(w + 7i)(w - 7i) = 0$

IV. $w + 7i = 0$; $w - 7i = 0$

$w = \pm 7i$

Section 6-8

The Complex Numbers

Objectives

- to find sums and differences of complex numbers
- to find product and quotients of complex numbers

Properties of Complex Numbers

- When adding complex numbers add the real portions together and add the imaginary portions together to create a new complex number.
- When multiplying complex numbers use the FOIL-ing process.
- Remember that an imaginary number is really a radical and cannot remain in the denominator of a fraction. Use the conjugate of a complex number to rationalize complex quotients.

Examples #1, 15 & 29.

#1 $(9 + 2i) + (1 - 7i)$

$(9 + 1) + (2i - 7i)$

$10 - 5i$

#15 $(- 4 + i)(8 + 5i)$

$(- 4)(8) + (- 4)(5i) + (i)(8) + (i)(5i)$

$- 32 + (- 20i) + 8i + 5i^2$

$- 32 - 12i - 5$

$- 37 - 12i$

29 $\dfrac{5}{3+4i}$ $\qquad\qquad$ $\dfrac{5(3-4i)}{25}$

$\left(\dfrac{5}{3+4i}\right)\left(\dfrac{3-4i}{3-4i}\right)$ $\qquad\qquad$ $\dfrac{(3-4i)}{5} = \dfrac{3}{5} - \dfrac{4i}{5}$