Section 8-1

Direct Variation and Proportion

Objectives

- to solve problems using direct variation

Direct Variation

- A linear function defined by the form y = mx where m ≠ 0 is called a direct variation
- "y varies directly with x" or "y is directly proportional with x"
- m is called the constant of variation or m is the constant of proportionality
- in a direct variation: $\frac{y_1}{x_1} = \frac{y_2}{x_2}$

Example Homework Problem

- If p is directly proportional to r - 2, and p = 20 when r = 6, find p when r = 12.
- For all corresponding values of this variation:

$$\frac{p_1}{r_1 - 2} = \frac{p_2}{r_2 - 2}$$

- Therefore:

$$\frac{20}{6-2} = \frac{p}{12-2}$$

$$5 = \frac{p}{10}$$

$$\frac{20}{4} = \frac{p}{10}$$

p = 50

Section 8-2

Inverse and Joint Variation

Objectives

- to solve problems using inverse variation
- to solve problems using joint variation

Inverse Variation

- a function defined by the equation of the form $xy = k$ where $x \neq 0$ and $k \neq 0$ is called an inverse variation
- "y varies inversely with x" or "y is inversely proportional to x"
- k is the constant of variation or the constant of proportionality
- in an inverse variation: $x_1y_1 = x_2y_2$

Joint Variation

- When an equation takes the form z = kxy it is called a joint variation
- "z varies jointly with x and y" or "z is jointly proportional to x and y"
- k is the constant of proportion or k is the constant of proportionality
- in a joint variation: $\frac{z_1}{x_1 y_1} = \frac{z_2}{x_2 y_2}$

Example Homework Problem

- Suppose that z varies jointly as u and v and inversely as w, and that z = 0.8 when u = 8, v = 6 and w = 5. Find z when u = 3, v = 10 and w = 5
- For all corresponding values of this variation:

$$\frac{z_1 w_1}{u_1 v_1} = \frac{z_2 w_2}{u_2 v_2}$$

- Therefore:

$$\frac{(0.8)(5)}{(8)(6)} = \frac{(z)(5)}{(3)(10)}$$

$$\frac{(8)(5)(3)(10)}{(10)(8)(6)(5)} = z$$

$$\frac{(0.8)(5)(3)(10)}{(8)(6)(5)} = z$$

$$z = \frac{1}{2}$$

Section 8-3

Dividing Polynomials

Objectives

- to perform polynomial long division

Polynomial Long Division

- polynomial long division works the same as numerical long division
- $$\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}$$
- Dividend = (Quotient)(Divisor) + Remainder

Steps to perform polynomial long division

- Write both the dividend and the divisor in decreasing order by degree.
- If any powers of the variable are missing (in either the dividend or the divisor), then insert a term raised to that power with a coefficient of zero.
- Determine what factor must be multiplied by the first term in the divisor to make it equal the first term in the dividend. That factor will be the first term in the quotient.
- Multiply the term to all of the terms in the divisor and subtract from the dividend. Repeat this process until you can divide no longer.

Example of polynomial long division

$$\frac{x^4 - x^3 - 10x + 10}{x - 3} \qquad x - 3\overline{)x^4 - x^3 + 0x^2 - 10x + 10}$$

$$\begin{array}{r}
x^3 + 2x^2 + 6x + 8 \\
x - 3\overline{)x^4 - x^3 + 0x^2 - 10x + 10} \\
\underline{x^4 - 3x^3} \qquad\qquad\qquad\qquad \\
2x^3 + 0x^2 \qquad\qquad\quad \\
\underline{2x^3 - 6x^2} \qquad\qquad\quad \\
6x^2 - 10x \qquad\quad \\
\underline{6x^2 - 18x} \qquad\quad \\
8x + 10 \\
\underline{8x - 24} \\
34
\end{array}$$

$$x^3 + 2x^2 + 6x + 8 + \frac{34}{x - 3}$$

Example of polynomial long division

$$\frac{9x^4 + 6x^3 + 4x + 4}{3x^2 + 2x + 2}$$

$$3x^2 + 2x + 2 \overline{\big) 9x^4 + 6x^3 + 0x^2 + 4x + 4}$$

$$\begin{array}{rl}
 & 3x^2 - 2 \\
3x^2 + 2x + 2 \big) & \overline{9x^4 + 6x^3 + 0x^2 + 4x + 4} \\
 & \underline{9x^4 + 6x^3 + 6x^2} \\
 & -6x^2 + 4x + 4 \\
 & \underline{-6x^2 - 4x - 4} \\
 & 8x + 8
\end{array}$$

$$3x^2 - 2 + \frac{8x + 8}{3x^2 + 2x + 2}$$

Example of polynomial long division with two variables

$$\frac{x^6 - a^6}{x^2 + ax + a^2}$$

$$
\begin{array}{r}
x^4 - ax^3 + a^3x - a^4 \\
x^2 + ax + a^2 \,\big)\, \overline{x^6 + 0ax^5 + 0a^2x^4 + 0a^3x^3 + 0a^4x^2 + 0a^5x - a^6} \\
\underline{x^6 + ax^5 + a^2x^4} \\
-ax^5 - a^2x^4 + 0a^3x^3 \\
\underline{-ax^5 - a^2x^4 - a^3x^3} \\
a^3x^3 + 0a^4x^2 + 0a^5x \\
\underline{a^3x^3 + a^4x^2 + a^5x} \\
-a^4x^2 - a^5x - a^6 \\
\underline{-a^4x^2 - a^5x - a^6} \\
0
\end{array}
$$

$x^4 - ax^3 + a^3x - a^4$

Section 8-4

Synthetic Division

Objectives

- to perform polynomial synthetic division

Synthetic Division

- Synthetic division is simply an algebraic version of short division.
- Polynomials must still be put in decreasing order by degree for both the dividend and the divisor.
- Missing terms must still be filled in with terms that have a zero coefficient for both the dividend and the divisor.
- In addition, the divisor must have the coefficient of x factored out before synthetic division is performed and then it must be redistributed to the quotient and the remainder.

Example of synthetic division

$$\frac{2x^4 - 7x^3 + 7x + 6}{x - 3}$$

3	2	- 7	0	7	6
		6	- 3	- 9	- 6
	2	- 1	- 3	- 2	0

$2x^3 - x^2 - 3x - 2$

Example of synthetic division

$$\frac{2x^3 - 3x^2 + 4x - 2}{2x + 1}$$

$$\frac{1}{2}\left(\frac{2x^3 - 3x^2 + 4x - 2}{x + \frac{1}{2}}\right)$$

$-\frac{1}{2}$	2	-3	4	-2
		-1	2	-3
	2	-4	6	-5

$$\frac{1}{2}\left(2x^2 - 4x + 6 + \frac{-5}{x + \frac{1}{2}}\right)$$

$$x^2 - 2x + 3 + \frac{-5}{2x + 1}$$

Section 8-5

The Remainder and Factor Theorems

Objectives

- to use synthetic substitution to find the value of a polynomial for a given value of the domain
- to determine whether a binomial is a factor using synthetic substitution
- to solve a polynomial equation using synthetic substitution
- to write a polynomial equation given its roots

Remainder and Factor Theorems

- Remainder Theorem: Let P(x) be a polynomial of positive degree n. Then for any number c, $P(x) = Q(x) \bullet (x - c) + P(c)$ where Q(x) is a polynomial with degree n - 1.
- Factor Theorem: The polynomial P(x) has x - r as a factor if and only if r is a root of the equation $P(x) = 0$.

Steps for homework problems

- Problems 1-8: perform synthetic division using the value of c.
- Problems 9-16: perform synthetic division with the value that makes the binomial zero. If the remainder is zero answer “yes”, if the remainder isn’t zero answer “no”.
- Problems 17-20: perform synthetic division with the given value and then factor or use the quadratic formula to break down the quotient polynomial.
- Problems 21-28: write each number a s a factor and then multiply the binomials together.
- Problems 29-32: perform synthetic division with the first value given, then perform synthetic division on the quotient with the second value given. Either factor or use the quadratic formula on the final quotient to find last roots.

Steps for homework problems

- Problems 33 & 34: perform synthetic division with the given value and show that the remainder is zero. Perform synthetic division again on the quotient with the same value and show that the remainder is again zero.
- Problems 35-37: perform synthetic division with given value of r until you no longer get zero as a remainder.

Example for Problems 1-8

$P(x) = x^3 - 2x^2 - 5x - 7; \quad c = 4$

4	1	- 2	- 5	- 7
		4	8	12
	1	2	3	5

P(4) = 5

Example for Problems 9-16

x + 1; P(x) $x^7 - x^5 + x^3 - x$

-1	1	0	-1	0	1	0	-1	0
		-1	1	0	0	-1	1	0
	1	-1	0	0	1	-1	0	0

yes

Example for Problems 17-20

$x^3 + 3x^2 - 3x - 9 = 0; -3$

-3	1	3	-3	-9
		-3	0	9
	1	0	-3	0

$x^2 - 3 = 0$

$x = -3, \pm\sqrt{3}$

Example for Problems 21-28

1, 2, - 3

$$(x - 1)(x - 2)(x + 3) = 0$$

$$(x^2 - 3x + 2)(x + 3) = 0$$

$$x^3 - 3x^2 + 2x + 3x^2 - 9x + 6 = 0$$

$$\mathbf{x^3 - 7x + 6 = 0}$$

Example for Problems 29-32

$$x^4 - 3x^3 - 8x^2 + 12x + 16 = 0;\ -1, 4$$

-1	1	-3	-8	12	16
		-1	4	4	-16
4	1	-4	-4	16	0
		4	0	-16	
	1	0	-4	0	

$$x^2 - 4 = 0$$

x = - 1, 4, 2, - 2

Example for Problems 33 & 34

$P(x) = x^4 + 2x^2 + 8x + 5$; $r = -1$

-1	1	0	2	8	5
		-1	1	-3	-5
-1	1	-1	3	5	0
		-1	2	-5	
	1	-2	5	0	

$$\frac{2 \pm \sqrt{4-(4)(1)(5)}}{2} = 1 \pm 2i$$

Example for Problems 35-37

$P(x) = x^4 + 4x^3 - 16x - 16;\ r = -2$

- 2	1	4	0	- 16	- 16
		- 2	- 4	8	16
- 2	1	2	- 4	- 8	0
		- 2	0	8	
- 2	1	0	- 4	0	
		- 2	4		
	1	- 2	0		

$x - 2 = 0;\ x = 2$ **triple root**

Section 8-6

Conjugate Root Theorem and Descartes Rule of Signs

Objectives

- to write a polynomial equation given its roots
- to solve a polynomial equation using synthetic substitution
- to determine the possible roots using Descartes rule of signs
- to determine if a number is root of a polynomial equation by using synthetic substitution

Theorems about Roots

- Number of Roots Theorem: Every polynomial with complex coefficients and positive degree n has exactly n roots.
- Conjugate Root Theorem: If a polynomial equation with real coefficients has a + bi as a root (a & $b \in \Re$, $b \neq 0$), then a - bi is also a root.
- Descartes' Rule of Signs: Let P(x) be a simplified polynomial with real coefficients and terms arranged in decreasing degree of x.
  - The number of positive real roots of P(x) = 0 equals the number of variations of sign of P(x) or is fewer than this number by an even integer.
  - The number of negative real roots of P(x) = 0 equals the number of variations of sign of P(- x) or is fewer than this number by an even integer.

Steps for Homework Problems

- Problems 1-4: write each number as a factor and the conjugate of any complex number as a factor, then multiply the factors together.
- Problems 5-8: Write the conjugate of the complex root.
- Problems 9-12: synthetically divide by the given complex number and its conjugate. There is an easier way to do this problem if you under stand polynomial long division.
- Problems 13-20: use Descartes' Rule of Signs to determine the possible number of each type of roots.
- Problems 21 & 22: Write the given numbers and their conjugates as factors. Multiply these factors together.
- Problems; 23 & 24: Perform synthetic division with the given complex number and its conjugate.
- Problem 26: use Descartes' Rule of Signs to help you generate a table for all possible roots.

Example for Problems 1-4

$$-1, 5i, -5i$$

$$(x + 1)(x - 5i)(x + 5i) = 0$$

$$(x + 1)(x^2 + 25) = 0$$

$$\mathbf{x^3 + x^2 + 25x + 25 = 0}$$

Example for Problems 5-8

$x^3 - 3x^2 + 4x - 12 = 0$; 3 and 2i

- 2i

Example for Problems 9-12

$x^3 + x - 10 = 0;\ -1 + 2i$

$-1 + 2i$	1	0	1	-10
		$-1 + 2i$	$-3 - 4i$	10
$-1 - 2i$	1	$-1 + 2i$	$-2 - 4i$	0
		$-1 - 2i$	$2 + 4i$	
	1	-2	0	

$x - 2 = 0$

$x = -1 + 2i,\ -1 - 2i,\ 2$

Example for Problems 13-20

- $x^4 + 2x^3 + x^2 + 1 = 0$
- If x is positive then the terms are: (+) (+) (+) (+). There are no sign changes so there are no positive real roots.
- If x is negative then the terms are (+) (-) (+) (+). There are two sign changes so there are either 2 negative real roots or no negative real roots.
- **The polynomial is degree 4 so there are 4 roots:**
  - **no positive real roots**
  - **two negative real root**
  - **two imaginary roots**

  OR
  - **no positive real roots**
  - **no negative real roots**
  - **four imaginary roots**

Example for Problems 21 & 22

$$2i,\ 1 - i,\ -2i,\ 1 + i$$

$$(x - 2i)(x - 1 + i)(x + 2i)(x - 1 - i) = 0$$

$$(x^2 + 4)(x^2 - 2x + 2) = 0$$

$$x^4 - 2x^3 + 2x^2 + 4x^2 - 8x + 8 = 0$$

$$\mathbf{x^4 - 2x^3 + 6x^2 - 8x + 8 = 0}$$

Example for Problems 23 & 24

$x^3 + 2x^2 + x - 1 + i = 0; \ -1 + i$

-1 + i	**1**	**2**	**1**	**-1 + i**
		-1 + i	**-2**	**1 - i**
-1 - i	**1**	**1 + i**	**-1**	**0**
		-1 - i	**0**	
	1	**0**	**-1**	

Section 8-7

Finding Rational Roots

Objectives

- to find the rational roots of a polynomial equation

Rational Root Theorem

- A polynomial equation with integral coefficients has the root $\frac{h}{k}$ where h and k are relatively prime integers. Then h must be a factor of the constant term and k must be a factor of the coefficient of the highest degree term.

Steps for the homework

- Use Descartes' Rule of signs to find the number and types of possible roots. In problems 22-25, first multiply to get rid of the fractions.
- Use the Rational Root Theorem to determine all of the possible rational roots.
- Use synthetic division to identify the rational roots of the equation, if any. If you find one rational root then you must do the next step. If you don't find one rational root then you may stop and your answer will be **"no rational roots"**.
- Use factoring or the quadratic formula to find the remaining roots.

Example for the homework

- $2x^3 - 3x^2 + 2x + 2 = 0$. When x is (+) there are two sign changes; therefore, there are either 2 or 0 positive real roots. When x is (-) there is one sign change; therefore, there is one negative real root.
- rational root theorem: $\frac{\pm 1 \pm 2}{\pm 1 \pm 2} = \left\{ \pm 1 \pm 2 \pm \frac{1}{2} \right\}$
-

- ½	2	- 3	2	2
		- 1	2	- 2
	2	- 4	4	0

Start synthetic division with the negative numbers because there is exactly one according to Descartes. There might not be any positive numbers.

- $2x^2 - 4x + 4 = 0$: has a GCF of 2
- $x^2 - 2x + 2 = 0$: does not factor

$$\frac{2 \pm \sqrt{4-(4)(1)(2)}}{2} = 1 \pm i$$

- $x = \left\{-\frac{1}{2}, 1+i, 1-i\right\}$

Section 8-8

Approximating Irrational Roots

Objectives

- to approximate to the nearest half unit the irrational roots of a polynomial equation

Intermediate Value Theorem

- If P is a polynomial function with real coefficients, and m is any number between P(a) and P(b), then there is at least one number c between a and b for which P(c) = m.

Steps for homework

- Perform Descartes’ Rule of Signs to get an idea of where the roots might be.
- To search out the irrational roots of the polynomials synthetically divide by all integers from - 4 to 4.
- Look for sign changes in the remainders to indicate that the graph has passed across the x-axis.
- Synthetically divide by any half unit near a possible root.
- Choose the value of x that has the y value closest to 0.

Example for Problems 1-10

- $P(x) = x^4 - 3x^3 + 5$: There are either 2 or 0 positive real roots, 0 negative real roots, and 2 or 4 imaginary roots; therefore, you would only synthetically divide the integers from 0 to 4.

1	1	-3	0	0	5
		1	-2	-2	-2
	1	-2	-2	-2	3

3	1	-3	0	0	5
		3	0	0	0
	1	0	0	0	5

2	1	-3	0	0	5
		2	-2	-4	-8
	1	-1	-2	-4	-3

Since I have two sign changes I know I have located the relative position of the two positive roots. Now I have to check the half units to see if they are closer.

$$
\begin{array}{c|ccccc}
1 & 1 & -3 & 0 & 0 & 5 \\
 & & 1 & -2 & -2 & -2 \\
\hline
 & 1 & -2 & -2 & -2 & \vert\ 3
\end{array}
$$

$$
\begin{array}{c|ccccc}
2 & 1 & -3 & 0 & 0 & 5 \\
 & & 2 & -2 & -4 & -8 \\
\hline
 & 1 & -1 & -2 & -4 & \vert\ -3
\end{array}
$$

$$
\begin{array}{c|ccccc}
3 & 1 & -3 & 0 & 0 & 5 \\
 & & 3 & 0 & 0 & 0 \\
\hline
 & 1 & 0 & 0 & 0 & \vert\ 5
\end{array}
$$

$$
\begin{array}{c|ccccc}
\frac{3}{2} & 1 & -3 & 0 & 0 & 5 \\
 & & \frac{3}{2} & -\frac{9}{4} & -\frac{27}{8} & -\frac{81}{16} \\
\hline
 & 1 & -\frac{3}{2} & -\frac{9}{4} & -\frac{27}{8} & \vert\ -\frac{1}{16}
\end{array}
$$

$$
\begin{array}{c|ccccc}
\frac{5}{2} & 1 & -3 & 0 & 0 & 5 \\
 & & \frac{5}{2} & -\frac{5}{4} & -\frac{25}{8} & -\frac{125}{16} \\
\hline
 & 1 & -\frac{1}{2} & -\frac{5}{4} & -\frac{25}{8} & \vert\ -\frac{45}{16}
\end{array}
$$

$$x \approx \frac{5}{2} \text{ and } \frac{3}{2}$$

Section 8-9

Linear Interpolation

Objectives

- to approximate values through the process of linear interpolation

Linear Interpolation

- If two points are very close to each other on a curve, then the difference between a straight line connecting these points and the curve connecting these points will be minimal.
- It is therefore possible to give estimates for values from the curve by using the values from the straight line.
- This process of estimation is called linear interpolation.
- To perform linear interpolation you must set up a proportion of corresponding values.

Example Homework Problem

- Approximate the population of the United States in 1915.
- Since the census is only completed every ten years this data is not directly available. We have the populations for (1910, 92 million) and (1920, 106 million). We will use these values to create a proportion.

year	pop.
1910	92
1915	y
1920	106

10 (1910–1920)

5 (1910–1915)

Δy (92 to y)

14 (92 to 106)

$$\frac{5}{10} = \frac{\Delta y}{14}$$

$$\frac{1}{2} = \frac{\Delta y}{14}$$

$$\Delta y = 7$$

y ≈ 99 million