Section 16-1

Definitions of Terms

Objectives

- to write a zero matrix
- to find the values of missing elements in a matrix equation

Matrices

- matrix: a rectangular array of numbers enclosed with brackets, named with a capital letter
- elements: the numbers in a matrix
- index of an element: an ordered pair (row, column) of numbers representing the position of the element in the array
- dimensions: (number of rows) x (number of columns)
- row matrix: an array with a single row
- column matrix: an array with a single column
- square matrix: an array with equal numbers of rows and columns or an (n x n) matrix
- zero matrix ($O_{m \times n}$): every element in the array is zero
- equal matrices: two matrices are equal if and only if they have the same dimensions and the elements in all corresponding positions are equal.

Example for 1-4

Write $O_{2 \times 5}$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example for 5-18

$$\begin{bmatrix} 2x + 3y \\ x - y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$2x + 3y = 3$
$x - y = 4$

$2x + 3y = 3$
$x = y + 4$

$2(y + 4) + 3y = 3$

$2y + 8 + 3y = 3$

$5y = -5$

$y = -1$

$x = -1 + 4$

$x = 3$

(3, - 1)

Example for 19-22

$$\begin{bmatrix} ax+b & cy+d \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

ax + b = 1 & cy + d = 0

ax = 1 - b & cy = - d

$$x = \frac{1-b}{a} \quad \& \quad y = -\frac{d}{c}$$

Section 16-2

Addition and Scalar Multiplication

Objectives

- to perform matrix addition and subtraction
- to perform matrix scalar multiplication
- to solve a matrix equation for a missing matrix

Arithmetic Properties of Matrices

- For each set of m x n matrices, $O_{m \times n}$ is the identity for addition. $A_{m \times n} + O_{m \times n} = O_{m \times n} + A_{m \times n} = A_{m \times n}$
- For each set of m x n matrices, the additive inverse of A is the matrix - A. $A_{m \times n} +(- A_{m \times n}) = - A_{m \times n} + A_{m \times n} = O_{m \times n}$
- For each set of m x n matrices, subtraction is defined as follows: $A_{m \times n} - B_{m \times n} = A_{m \times n} + (- B_{m \times n})$

Let A, B and C be m x n matrices. Let $O_{m \times n}$ be the m x n zero matrix.

- Closure Property: A + B is an m x n matrix
- Commutative Property: $A + B = B + A$
- Associative Property: $(A + B) + C = A + (B + C)$
- Identity Property: $A + O_{m \times n} = O_{m \times n} + A = A$
- Inverse Property: $A + (- A) = - A + A = O_{m \times n}$

Properties of Scalar Multiplication

Let A, B and C be m x n matrices. Let $O_{m \times n}$ be the m x n zero matrix, and let p and q be scalars.

- Closure Property: pA is an m x n matrix
- Commutative Property: pA = Ap
- Associative Property: p(qA) = (pq)A
- Distributive Property: (p + q)A = pA + qA
  p(A+ B) = pA + pB
- Identity Property: 1(A) = A
- Multiplicative Property of - 1: (- 1)A = - A
- Multiplicative Property of 0: 0(A) = $O_{m \times n}$

Example for 1-16

$$2\begin{bmatrix} 5 & -2 \\ -3 & 4 \\ 0 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 7 \\ 0 & -4 \\ 6 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 10 & -4 \\ -6 & 8 \\ 0 & 12 \end{bmatrix} + \begin{bmatrix} 1 & 7 \\ 0 & -4 \\ 6 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 11 & 3 \\ -6 & 4 \\ 6 & 17 \end{bmatrix}$$

Example for 17-20

$$X + 3\begin{bmatrix} -3 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -4 & 10 \\ 12 & 0 \end{bmatrix}$$

$$X + \begin{bmatrix} -9 & 6 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -4 & 10 \\ 12 & 0 \end{bmatrix}$$

$$X + \begin{bmatrix} -9 & 6 \\ 0 & -3 \end{bmatrix} - \begin{bmatrix} -9 & 6 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -4 & 10 \\ 12 & 0 \end{bmatrix} - \begin{bmatrix} -9 & 6 \\ 0 & -3 \end{bmatrix}$$

$$X = \begin{bmatrix} -4 & 10 \\ 12 & 0 \end{bmatrix} - \begin{bmatrix} -9 & 6 \\ 0 & -3 \end{bmatrix}$$

$$X = \begin{bmatrix} 5 & 4 \\ 12 & 3 \end{bmatrix}$$

Section 16-3

Matrix Multiplication

Objectives

- to perform matrix multiplication

Matrix Multiplication

- The product of matrices $A_{m \times n}$ and $B_{n \times p}$ is the m x p matrix whose element in the a^{th} row b^{th} column is the sum of the products of corresponding elements of the a^{th} row of A and the b^{th} column of B.
- Two matrices can be multiplied only if the number of columns in the first matrix is equal to the number of rows in the second matrix.
- identity matrix($I_{n \times n}$): a square matrix whose main diagonal, from element (1,1) to (n, n), has all 1's and all other elements are 0.

Properties of Matrix Multiplication

- For any n x n matrix A,
  $I_{n \times n} \cdot A = A \cdot I_{n \times n} = A$
- If A is any n x n matrix and $O_{n \times n}$ is the zero matrix,
  $O_{n \times n} \cdot A = A \cdot O_{n \times n} = O_{n \times n}$

Let A, B and C be n x n matrices. Let $I_{n \times n}$ be the identity matrix and $O_{n \times n}$ be the zero matrix.

- Associative Property: $(AB)C = A(BC)$
- Distributive Property: $A(B + C) = AB + AC$
  $(B + C)A = BA + CA$
- Identity Property: $I_{n \times n} \cdot A = A \cdot I_{n \times n} = A$
- Multiplicative Property of $O_{n \times n}$: $O_{n \times n} \cdot A = A \cdot O_{n \times n} = O_{n \times n}$
- **There is no Commutative Property for matrix multiplication. If you change the order you change the answer.**

Example for 1-12

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 1 & 0 \end{bmatrix}$$

A 3x2 multiplied to a 2x4 creates a 3x4 matrix.

$$\begin{bmatrix} e_{1,1} & e_{1,2} & e_{1,3} & e_{1,4} \\ e_{2,1} & e_{2,2} & e_{2,3} & e_{2,4} \\ e_{3,1} & e_{3,2} & e_{3,3} & e_{3,4} \end{bmatrix} = \begin{bmatrix} -1 & 2 & 2 & -1 \\ 4 & -1 & -1 & -3 \\ -4 & 4 & 4 & 0 \end{bmatrix}$$

$e_{1,1} = (1)(1) + (2)(-1) = -1$ $e_{2,1} = (3)(1) + (-1)(-1) = 4$ $e_{3,1} = (0)(1) + (4)(-1) = -4$

$e_{1,2} = (1)(0) + (2)(1) = 2$ $e_{2,2} = (3)(0) + (-1)(1) = -1$ $e_{3,2} = (0)(0) + (4)(1) = 4$

$e_{1,3} = (1)(0) + (2)(1) = 2$ $e_{2,3} = (3)(0) + (-1)(1) = -1$ $e_{3,3} = (0)(0) + (4)(1) = 4$

$e_{1,4} = (1)(-1) + (2)(0) = -1$ $e_{2,4} = (3)(-1) + (-1)(0) = -3$ $e_{3,4} = (0)(-1) + (4)(0) = 0$

Example for 13-20

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \; C = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

Find (A + B)(A - B) and $A^2 - B^2$.

$$\left(\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\right)\left(\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\right)$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\right) - \left(\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\right)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

Notice (A + B)(A - B) $\neq$ $A^2 - B^2$.

Example for 21-26

Find$(-D)^2$. $D = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ 1 & 0 & 0 \end{bmatrix}$

$$-D = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$(-D)^2 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$(-D)^2 = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 4 & -2 \\ 1 & 0 & -1 \end{bmatrix}$$

Section 16-4

Applications of Matrices

Objectives

- to work with applying matrices to communication networks and dominance relations

Communication Networks & Dominance Relationships

- communication networks and dominance relationships are represented by square matrices where the number of rows and columns is determined by either the number of communicators or the number of teams
- in both types of problems no communication or a loss is represented by a zero in the matrix, zeroes are always placed in the matrix in the position where an entity is communicating with itself or where a team is playing itself
- in both types of problems communication or a win is represented by a one in the matrix
- squaring the matrix creates a new matrix that either represents communication through one relay or 2nd level dominance; likewise, cubing the matrix creates a new matrix that either represents communication through two relays or 3rd level dominance

Examples of Problems

- Read the two examples in the section. They do a good job of illustrating in depth how communication networks and dominance relationships work.

Section 16-5

Determinants

Objectives

- to evaluate the determinant of a matrix

Determinant of a Matrix

- determinant: a real number associated with a square matrix
- order of a square matrix: either the number of rows or the number of columns
- Determinant of a 2x2 matrix (2^{nd} order determinant):

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } \det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- Determinant of a 3x3 matrix (3^{rd} order determinant):

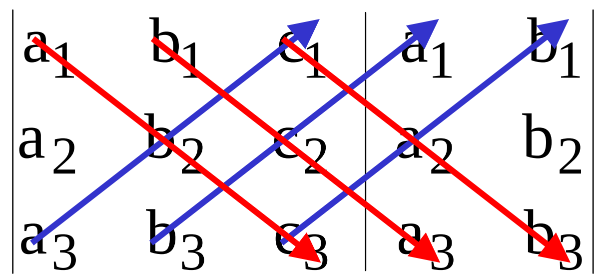
$$\text{Let } B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \text{ then } \det B = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$$

$$a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2$$

Calculating a 3rd Order Determinant

$$\det B = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Copy the first two columns onto the end of the matrix.



Multiply diagonally down through the matrix adding the products.

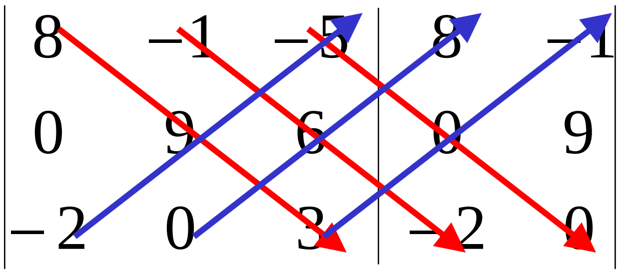
$$a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1$$

Multiply diagonally up through the matrix subtracting the products.

$$a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3$$

Example for 1-14

$$\begin{vmatrix} 8 & -1 & -5 \\ 0 & 9 & 6 \\ -2 & 0 & 3 \end{vmatrix}$$



(8)(9)(3) + **(- 1)(6)(- 2)** + **(- 5)(0)(0)**

(8)(9)(3) + **(- 1)(6)(- 2)** + **(- 5)(0)(0)** - **(- 2)(9)(- 5)** - **(0)(6)(8)** - **(3)(0)(- 1)**

(216) + **(12)** + **(0)** - **(90)** - **(0)** - **(0)**

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Example for 15-18

Show that for any 2x2 matrix, A, if all the elements in one row or one column are zeroes, then det A = 0.

$$A = \begin{bmatrix} 0 & 0 \\ x & y \end{bmatrix}$$

$$\det A = \begin{vmatrix} 0 & 0 \\ x & y \end{vmatrix} = (0)(y) - (x)(0) = 0 - 0 = 0$$

Section 16-6

Inverses of Matrices

Objectives

- to find the inverse, if any, of a matrix
- to use a matrix to solve a system
- to solve a matrix equation

Inverse of a Matrix

- inverse matrices: for any two matrices A and B, if AB = BA = I, then A and B are inverse matrices and $B = A^{-1}$ and $A = B^{-1}$.
- Inverses of a 2x2 Matrix:

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } \det A \neq 0, \text{ then } A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If det A = 0, then A has no inverse.

Example for 1-8

$$A = \begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 3 & -1 \\ 6 & 2 \end{vmatrix} = (3)(2) - (6)(-1) = 6 + 6 = 12$$

$$A^{-1} = \frac{1}{12} \begin{bmatrix} 2 & 1 \\ -6 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{2}{12} & \frac{1}{12} \\ \frac{-6}{12} & \frac{3}{12} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{6} & \frac{1}{12} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

Example for 9-16

2x - y = 6

3x + 2y = - 19

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -19 \end{bmatrix}$$

$$\frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -19 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -7 \\ -56 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ -8 \end{bmatrix}$$

(- 1, -8)

Example for 17-20

$$\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} X - \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} X - \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{3}{2} \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{3}{2} \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} X = \begin{bmatrix} 2 & \frac{7}{2} \\ -1 & \frac{5}{2} \end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} X = \frac{1}{2}\begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 & \frac{7}{2} \\ -1 & \frac{5}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X = \frac{1}{2}\begin{bmatrix} 4 & -\frac{3}{2} \\ -6 & \frac{13}{2} \end{bmatrix} \qquad X = \begin{bmatrix} 2 & -\frac{3}{4} \\ -3 & \frac{13}{4} \end{bmatrix}$$

Section 16-7

Expansion of Determinants by Minors

Objectives

- to perform expansion by minors to calculate the determinant of a matrix

Expansion of Determinants by Minors

- minor of an element: is the determinant resulting from the deletion of one row and one column containing the element
- this process can be used on determinant of the 3rd or higher order
- expanding by minors:
  - multiply each element of a given row or column by its minor
  - add the numbers of the index of the elements, if the sum is odd subtract the product from the determinant, if the sum is even add the product to the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Example for 1-8

$$\begin{vmatrix} 2 & 1 & 3 \\ -2 & 1 & 4 \\ 1 & 2 & 5 \end{vmatrix}$$

row one

$$2\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} - 1\begin{vmatrix} -2 & 4 \\ 1 & 5 \end{vmatrix} + 3\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$2(5 - 8) - 1(-10 - 4) + 3(-4 - 1)$$

$$2(-3) - 1(-14) + 3(-5)$$

$$-6 + 14 - 15$$

$$-7$$

Example for 9-12

$$\begin{vmatrix} 0 & 3 & 1 \\ 4 & -1 & 2 \\ 0 & 2 & 0 \end{vmatrix}$$

Expand along the 3rd row to take advantage of the two zeroes/

$$0\begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} + 0\begin{vmatrix} 0 & 3 \\ 4 & -1 \end{vmatrix}$$

0 - 2(0 - 4) + 0

- 2(- 4)

8

Example for 13-16

$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ x & 1 & 2 \end{vmatrix} = 4$$

Expand along the 3rd row.

$$x\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4$$

$$x(1 - 2) - 1(2 - 1) + 2(4 - 1) = 4$$

$$x(-1) - 1(1) + 2(3) = 4$$

$$-x + 5 = 4$$

$$\mathbf{x = 1}$$

Example for 17&18

$$\begin{vmatrix} 5 & 0 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ 8 & 2 & 2 & 0 \\ -4 & 10 & 3 & 2 \end{vmatrix}$$

Expand the 4th column to take advantage of the 3 zeroes.

$$-0\begin{vmatrix} 3 & -3 & 0 \\ 8 & 2 & 2 \\ -4 & 10 & 3 \end{vmatrix} + 0\begin{vmatrix} 5 & 0 & 0 \\ 8 & 2 & 2 \\ -4 & 10 & 3 \end{vmatrix} - 0\begin{vmatrix} 5 & 0 & 0 \\ 3 & -3 & 0 \\ -4 & 10 & 3 \end{vmatrix} + 2\begin{vmatrix} 5 & 0 & 0 \\ 3 & -3 & 0 \\ 8 & 2 & 2 \end{vmatrix}$$

$$2\begin{vmatrix} 5 & 0 & 0 \\ 3 & -3 & 0 \\ 8 & 2 & 2 \end{vmatrix}$$

Expand the 3rd column to take advantage of the 2 zeroes.

$$2\left(0\begin{vmatrix} 3 & -3 \\ 8 & 2 \end{vmatrix} - 0\begin{vmatrix} 5 & 0 \\ 8 & 2 \end{vmatrix} + 2\begin{vmatrix} 5 & 0 \\ 3 & -3 \end{vmatrix} \right)$$

2[2(- 15 - 0)]

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Section 16-8

Properties of Determinants

Objectives

- to use the properties of matrices to find the simplest way to a determinant

Properties of Determinants

- Property 1: If each element in any row or column is 0, then the determinant is equal to 0.
- Property 2: If two rows or columns of a determinant have corresponding elements that are equal, then the determinant is 0.
- Property 3: If two rows or columns of a determinant are interchanged, then the resulting determinant is the opposite of the original determinant.
- Property 4: If each element in one row or column of a determinant is multiplied by a real number k, then the determinant is multiplied by k.
- Property 5: If each element of one row or column is multiplied by a real number k and if the resulting products are then added to the corresponding elements of another row or column, then the resulting determinant equals the original one.

Example for 1-12

Use property 5 and add the 3rd row to the 2nd to make the matrix:

$$\begin{vmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 \end{vmatrix}$$

Expand along the 2nd row to take advantage of the 3 zeroes.

$$-2\begin{vmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{vmatrix}$$

Use property 5 again and add the 3rd column to the 2nd to make the matrix:

$$-2\begin{vmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & -1 \end{vmatrix}$$

Use property 1 to say that the determinant = **0**

Section 16-9

Cramer’s Rule

Objectives

- to use Cramer's Rule to solve a system
- to determine the number of solutions to a system

Cramer's Rule

- The solution of a system of n linear equations in n variables is given by

$$x = \frac{D_x}{D},\ \ y = \frac{D_y}{D}, \ldots$$

where D is the determinant of the matrix of coefficients of the variables ($D \neq 0$) and D_x, D_y, ... are derived from D by replacing the coefficients of x, y, ... respectively, by the constants.

- When $D = 0$ the system may have no solution or it may have infinitely many solutions.
- If $D = 0$ and $D_y \neq 0$, then the equations are inconsistent and the equations are parallel.
- If $D = 0$ and $D_y = 0$, then the equations are dependent and their graphs coincide.

Example for 1-10

$$2x + y - z = 2$$
$$x + y + z = 7$$
$$x + 2y + z = 4$$

$$D = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} \quad D_x = \begin{vmatrix} 2 & 1 & -1 \\ 7 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix} \quad D_y = \begin{vmatrix} 2 & 2 & -1 \\ 1 & 7 & 1 \\ 1 & 4 & 1 \end{vmatrix} \quad D_z = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 1 & 7 \\ 1 & 2 & 4 \end{vmatrix}$$

$D = - 3$ $\quad D_x = - 15$ $\quad D_y = 9$ $\quad D_z = - 15$

$$x = \frac{-15}{-3} \quad y = \frac{9}{-3} \quad z = \frac{-15}{-3}$$

(5, - 3, 5)

Example for 11-14

$$2x - y = 5$$
$$-2x + y = -1$$

$$D = \begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} = 0$$

$$D_y = \begin{vmatrix} 2 & 5 \\ -2 & -1 \end{vmatrix} = 8$$

Since D = 0 and $D_y \neq 0$, the graphs are inconsistent and there is **no solution**.

Example for 15&16

2a - 3b + c - 3d = - 1

a + 2b + 3c - d = 1

3a + 5b + 6c = 4

3a - b - 2d = 6

$$D_b = \begin{vmatrix} 2 & -1 & 1 & -3 \\ 1 & 1 & 3 & -1 \\ 3 & 4 & 6 & 0 \\ 3 & 6 & 0 & -2 \end{vmatrix} = 112$$

$$D = \begin{vmatrix} 2 & -3 & 1 & -3 \\ 1 & 2 & 3 & -1 \\ 3 & 5 & 6 & 0 \\ 3 & -1 & 0 & -2 \end{vmatrix} = 56$$

$$D_c = \begin{vmatrix} 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & -1 \\ 3 & 5 & 4 & 0 \\ 3 & -1 & 6 & -2 \end{vmatrix} = -112$$

$$a = \frac{112}{56}$$

$$b = \frac{112}{56}$$

$$c = \frac{-112}{56}$$

$$d = \frac{-56}{56}$$

$$D_a = \begin{vmatrix} -1 & -3 & 1 & -3 \\ 1 & 2 & 3 & -1 \\ 4 & 5 & 6 & 0 \\ 6 & -1 & 0 & -2 \end{vmatrix} = 112$$

$$D_d = \begin{vmatrix} 2 & -3 & 1 & -1 \\ 1 & 2 & 3 & 1 \\ 3 & 5 & 6 & 4 \\ 3 & -1 & 0 & 6 \end{vmatrix} = -56$$

(2, 2, - 2, - 1)