Surveyors use accurate measures of magnitudes and directions to create scaled maps of large regions.
Objectives: After completing this module, you should be able to:

- Demonstrate that you meet **mathematics expectations**: unit analysis, algebra, scientific notation, and right-triangle trigonometry.
- Define and give examples of **scalar** and **vector** quantities.
- Determine the **components** of a given vector.
- Find the **resultant** of two or more vectors.
Expectations

- You must be able convert units of measure for physical quantities.

Convert 40 m/s into kilometers per hour.

\[
\frac{40 \text{ m}}{\text{s}} \times \frac{1 \text{ km}}{1000 \text{ m}} \times \frac{3600 \text{ s}}{1 \text{ h}} = 144 \text{ km/h}
\]
Expectations (Continued):

• College algebra and simple formula manipulation are assumed.

Example: \[ x = \left( \frac{v_0 + v_f}{2} \right) t \]  
*Solve for \( v_o \)*

\[
v_0 = \frac{v_f t - 2x}{t}
\]
Expectations (Continued)

• You must be able to work in scientific notation.

Evaluate the following:

\[
F = \frac{Gmm'}{r^2} = \frac{(6.67 \times 10^{-11})(4 \times 10^{-3})(2)}{(8.77 \times 10^{-3})^2}
\]

\[
F = 6.94 \times 10^{-9} \text{ N} = 6.94 \text{ nN}
\]
Expectations (Continued)

- You must be familiar with SI prefixes

<table>
<thead>
<tr>
<th>Unit</th>
<th>Conversion</th>
</tr>
</thead>
<tbody>
<tr>
<td>The meter (m)</td>
<td></td>
</tr>
<tr>
<td>1 Gm</td>
<td>$1 \times 10^9$ m</td>
</tr>
<tr>
<td>1 Mm</td>
<td>$1 \times 10^6$ m</td>
</tr>
<tr>
<td>1 km</td>
<td>$1 \times 10^3$ m</td>
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<td></td>
<td></td>
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<tr>
<td>1 m</td>
<td>$1 \times 10^0$ m</td>
</tr>
<tr>
<td>1 nm</td>
<td>$1 \times 10^{-9}$ m</td>
</tr>
<tr>
<td>1 μm</td>
<td>$1 \times 10^{-6}$ m</td>
</tr>
<tr>
<td>1 mm</td>
<td>$1 \times 10^{-3}$ m</td>
</tr>
</tbody>
</table>
Expectations (Continued)

- You must have mastered right-triangle trigonometry.

\[
\begin{align*}
\sin \theta &= \frac{y}{R} \\
\cos \theta &= \frac{x}{R} \\
\tan \theta &= \frac{y}{x} \\
R^2 &= x^2 + y^2
\end{align*}
\]
If you feel you need to brush up on your mathematics skills, try the tutorial from Chap. 2 on Mathematics. Trig is reviewed along with vectors in this module.

Select Chap. 2 from the On-Line Learning Center in Tippens—Student Edition
Physics is the Science of Measurement

We begin with the measurement of length: its magnitude and its direction.
Distance: A Scalar Quantity

- **Distance** is the length of the actual path taken by an object.

A scalar quantity:

Contains **magnitude** only and consists of a **number** and a **unit**.

(20 m, 40 mi/h, 10 gal)
Displacement—A Vector Quantity

- **Displacement** is the straight-line separation of two points in a specified direction.

A vector quantity:

Contains magnitude **AND** direction, a number, unit & angle.

(12 m, 30°; 8 km/h, N)
Distance and Displacement

- **Displacement** is the \( x \) or \( y \) coordinate of position. Consider a car that travels 4 m, E then 6 m, W.

\[
\begin{align*}
D &= 2 \text{ m, W} \\
\text{Net displacement:} \\
\text{What is the distance traveled?} \\
10 \text{ m}!!
\end{align*}
\]
Identifying Direction

A common way of identifying direction is by reference to East, North, West, and South. (Locate points below.)

- 40 m, 50° N of E
- 40 m, 60° N of W
- 40 m, 60° W of S
- 40 m, 60° S of E

Length = 40 m
Identifying Direction

Write the angles shown below by using references to east, south, west, north.

- 50° S of E
- 45° W of N
Vectors and Polar Coordinates

Polar coordinates $(R, \theta)$ are an excellent way to express vectors. Consider the vector $40$ m, $50^\circ$ N of E, for example.

$R$ is the magnitude and $\theta$ is the direction.
Vectors and Polar Coordinates

Polar coordinates \((R, \theta)\) are given for each of four possible quadrants:

- \((R, \theta) = 40 \text{ m}, 50^\circ\)
- \((R, \theta) = 40 \text{ m}, 120^\circ\)
- \((R, \theta) = 40 \text{ m}, 210^\circ\)
- \((R, \theta) = 40 \text{ m}, 300^\circ\)
Rectangular Coordinates

Reference is made to $x$ and $y$ axes, with + and - numbers to indicate position in space.

Right, up = (+, +)
Left, down = (-, -)

$x,y) = (\text{?}, \text{?})$
Trigonometry Review

- Application of Trigonometry to Vectors

\[ y = R \sin \theta \]
\[ x = R \cos \theta \]
\[ R^2 = x^2 + y^2 \]
Example 1: Find the height of a building if it casts a shadow 90 m long and the indicated angle is 30°.

The height $h$ is opposite 30° and the known adjacent side is 90 m.

\[
\tan 30° = \frac{opp}{adj} = \frac{h}{90 \text{ m}}
\]

\[h = (90 \text{ m}) \tan 30°\]

\[h = 57.7 \text{ m}\]
Finding Components of Vectors

A component is the effect of a vector along other directions. The x and y components of the vector \((R, \theta)\) are illustrated below.

\[
x = R \cos \theta
\]

\[
y = R \sin \theta
\]

Finding components:

Polar to Rectangular Conversions
**Example 2:** A person walks 400 m in a direction of 30° N of E. How far is the displacement east and how far north?

The x-component (E) is ADJ: \[x = R \cos \theta\]

The y-component (N) is OPP: \[y = R \sin \theta\]
Example 2 (Cont.): A 400-m walk in a direction of $30^\circ$ N of E. How far is the displacement east and how far north?

Note: $x$ is the side adjacent to angle $30^\circ$

The $x$-component is:

$$x = R \cos \theta$$

$$x = (400 \text{ m}) \cos 30^\circ$$

$$= +346 \text{ m}, \text{ E}$$

$$R_x = +346 \text{ m}$$
Example 2 (Cont.): A 400-m walk in a direction of 30° N of E. How far is the displacement east and how far north?

Note: \( y \) is the side opposite to angle 30°

\[ \text{OPP} = \text{HYP} \times \sin 30° \]

\[ y = R \sin \theta \]

\[ y = (400 \text{ m}) \sin 30° \]

\[ = +200 \text{ m, N} \]

The \( y \)-component is:

\[ R_y = +200 \text{ m} \]
Example 2 (Cont.): A 400-m walk in a direction of 30° N of E. How far is the displacement east and how far north?

\[ R_x = +346 \text{ m} \]
\[ R_y = +200 \text{ m} \]

The x- and y-components are each + in the first quadrant.

Solution: The person is displaced 346 m east and 200 m north of the original position.
Signs for Rectangular Coordinates

First Quadrant:
- R is positive (+)
- $0^\circ < \theta < 90^\circ$
- $x = +$; $y = +$
- $x = R \cos \theta$
- $y = R \sin \theta$
Signs for Rectangular Coordinates

Second Quadrant:
- $R$ is positive (+)
- $90^\circ > \theta < 180^\circ$
- $x = -$; $y = +$

$x = R \cos \theta$
$y = R \sin \theta$
Third Quadrant:
R is positive (+)

\[ 180^\circ > \theta < 270^\circ \]

\[ x = - \]
\[ y = - \]

\[ x = R \cos \theta \]
\[ y = R \sin \theta \]
Fourth Quadrant:

- **R** is positive (+)
- $270^\circ > \theta < 360^\circ$
- $x = +$
- $y = -$

- $x = R \cos \theta$
- $y = R \sin \theta$
Resultant of Perpendicular Vectors

Finding resultant of two perpendicular vectors is like changing from rectangular to polar coord.

\[ R = \sqrt{x^2 + y^2} \]
\[ \tan \theta = \frac{y}{x} \]

R is always positive; \( \theta \) is from + x axis
Example 3: A 30-lb southward force and a 40-lb eastward force act on a donkey at the same time. What is the NET or resultant force on the donkey?

Draw a rough sketch. Choose rough scale:

Ex: 1 cm = 10 lb

Note: Force has direction just like length does. We can treat force vectors just as we have length vectors to find the resultant force. The procedure is the same!
Finding Resultant: (Cont.)

Finding \((R, \theta)\) from given \((x, y) = (+40, -30)\)

\[ R = \sqrt{x^2 + y^2} \]
\[ R = \sqrt{(40)^2 + (30)^2} = 50 \text{ lb} \]

\[ \tan \phi = \frac{-30}{40} \]
\[ \phi = -36.9^\circ \]

\[ \theta = 323.1^\circ \]
\[ \phi = 36.9^\circ; \quad \theta = 36.9^\circ; \quad 143.1^\circ; \quad 216.9^\circ; \quad 323.1^\circ \]
Unit vector notation \((i, j, k)\)

Consider 3D axes \((x, y, z)\)

Define unit vectors, \(i, j, k\)

Examples of Use:

- 40 m, E = 40 \(i\)
- 40 m, W = -40 \(i\)
- 30 m, N = 30 \(j\)
- 30 m, S = -30 \(j\)
- 20 m, out = 20 \(k\)
- 20 m, in = -20 \(k\)
Example 4: A woman walks 30 m, W; then 40 m, N. Write her displacement in $i, j$ notation and in $R, \theta$ notation.

In $i, j$ notation, we have:

$$R = R_x i + R_y j$$

$$R_x = -30 \text{ m} \quad R_y = +40 \text{ m}$$

$$R = -30i + 40j$$

Displacement is 30 m west and 40 m north of the starting position.
Example 4 (Cont.): Next we find her displacement in $R, \theta$ notation.

\[ R = \sqrt{(-30)^2 + (40)^2} \]

\[ R = 50 \text{ m} \]

\[ (R, \theta) = (50 \text{ m}, 126.9^\circ) \]

\[ \tan \phi = \begin{vmatrix} +40 \\ -30 \end{vmatrix} ; \quad \phi = 59.1^0 \]

\[ \theta = 180^0 - 59.1^0 \]

\[ \theta = 126.9^\circ \]
**Example 6:** Town A is 35 km south and 46 km west of Town B. Find length and direction of highway between towns.

\[ R = -46 \hat{i} - 35 \hat{j} \]

\[ R = \sqrt{(46 \text{ km})^2 + (35 \text{ km})^2} \]

\[ R = 57.8 \text{ km} \]

\[ \tan \phi = \frac{-46 \text{ km}}{-35 \text{ km}} \]

\[ \phi = 52.7^0 \text{ S. of W.} \]

\[ \theta = 180^0 + 52.7^0 \]

\[ \theta = 232.7^0 \]
Example 7. Find the components of the 240-N force exerted by the boy on the girl if his arm makes an angle of 28° with the ground.

\[ F_x = -(240 \text{ N}) \cos 28^\circ = -212 \text{ N} \]

\[ F_y = +(240 \text{ N}) \sin 28^\circ = +113 \text{ N} \]

Or in \( \mathbf{i}, \mathbf{j} \) notation:

\[ F = -(212 \text{ N}) \mathbf{i} + (113 \text{ N}) \mathbf{j} \]
Example 8. Find the components of a 300-N force acting along the handle of a lawn-mower. The angle with the ground is $32^\circ$.

\[ F_x = -(300 \text{ N}) \cos 32^\circ = -254 \text{ N} \]

\[ F_y = -(300 \text{ N}) \sin 32^\circ = -159 \text{ N} \]

Or in \(i,j\) notation:

\[ F = -(254 \text{ N})i - (159 \text{ N})j \]
1. Start at origin. Draw each vector to scale with tip of 1st to tail of 2nd, tip of 2nd to tail 3rd, and so on for others.

2. Draw resultant from origin to tip of last vector, noting the quadrant of the resultant.

3. Write each vector in $i,j$ notation.

4. Add vectors algebraically to get resultant in $i,j$ notation. Then convert to $(R, \theta)$. 
**Example 9.** A boat moves 2.0 km east then 4.0 km north, then 3.0 km west, and finally 2.0 km south. Find resultant displacement.

1. Start at origin. Draw each vector to scale with tip of 1st to tail of 2nd, tip of 2nd to tail 3rd, and so on for others.

2. Draw resultant from origin to tip of last vector, noting the quadrant of the resultant.

Note: The scale is approximate, but it is still clear that the resultant is in the fourth quadrant.
3. Write each vector in $i,j$ notation:

- $A = +2 \ i$
- $B = +4 \ j$
- $C = -3 \ i$
- $D = -2 \ j$

$$R = -1 \ i + 2 \ j$$

1 km, west and 2 km north of origin.

4. Add vectors $A,B,C,D$ algebraically to get resultant in $i,j$ notation.

5. Convert to $R,\theta$ notation

See next page.
Example 9 (Cont.)

Find resultant displacement.

Resultant Sum is:

\[ \mathbf{R} = -1 \mathbf{i} + 2 \mathbf{j} \]

Now, We Find \( \mathbf{R}, \theta \)

\[ R = \sqrt{(-1)^2 + (2)^2} = \sqrt{5} \]

\[ R = 2.24 \text{ km} \]

\[ \tan \phi = \frac{+2 \text{ km}}{-1 \text{ km}} \]

\[ \phi = 63.4^0 \text{ N or W} \]
Reminder of Significant Units:

For convenience, we follow the practice of assuming three (3) significant figures for all data in problems.

In the previous example, we assume that the distances are 2.00 km, 4.00 km, and 3.00 km.

Thus, the answer must be reported as:

\[ R = 2.24 \text{ km}, \ 63.4^0 \text{ N of W} \]
Significant Digits for Angles

Since a **tenth of a degree** can often be significant, sometimes a fourth digit is needed.

**Rule:** Write angles to the nearest tenth of a degree. See the two examples below:

\[ \theta = 36.9^\circ; 323.1^\circ \]
**Example 10:** Find $R, \theta$ for the three vector displacements below:

- $A = 5 \text{ m}, 0^0$
- $B = 2.1 \text{ m}, 20^0$
- $C = 0.5 \text{ m}, 90^0$

1. First draw vectors $A$, $B$, and $C$ to approximate scale and indicate angles. (Rough drawing)

2. Draw resultant from origin to tip of last vector; noting the quadrant of the resultant. ($R, \theta$)

3. Write each vector in $i,j$ notation. (Continued ...
### Example 10: Find $R, \theta$ for the three vector displacements below: (A table may help.)

For $i,j$ notation find $x,y$ components of each vector $A, B, C.$

<table>
<thead>
<tr>
<th>Vector</th>
<th>$\phi$</th>
<th>X-component (i)</th>
<th>Y-component (j)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A=5 m</td>
<td>0°</td>
<td>+ 5 m</td>
<td>0</td>
</tr>
<tr>
<td>B=2.1m</td>
<td>20°</td>
<td>+(2.1 m) \cos 20°</td>
<td>+(2.1 m) \sin 20°</td>
</tr>
<tr>
<td>C=.5 m</td>
<td>90°</td>
<td>0</td>
<td>+ 0.5 m</td>
</tr>
</tbody>
</table>

\[
R_x = A_x + B_x + C_x \\
R_y = A_y + B_y + C_y
\]
Example 10 (Cont.): Find i,j for three vectors: \( A = 5 \text{ m}, 0^\circ; \ B = 2.1 \text{ m}, 20^\circ; \ C = 0.5 \text{ m}, 90^\circ. \)

<table>
<thead>
<tr>
<th>X-component (i)</th>
<th>Y-component (j)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_x = +5.00 \text{ m} )</td>
<td>( A_y = 0 )</td>
</tr>
<tr>
<td>( B_x = +1.97 \text{ m} )</td>
<td>( B_y = +0.718 \text{ m} )</td>
</tr>
<tr>
<td>( C_x = 0 )</td>
<td>( C_y = +0.50 \text{ m} )</td>
</tr>
</tbody>
</table>

4. Add vectors to get resultant \( R \) in i,j notation.

\[
\begin{align*}
A &= 5.00 \text{ i} + 0 \text{ j} \\
B &= 1.97 \text{ i} + 0.718 \text{ j} \\
C &= 0 \text{ i} + 0.50 \text{ j} \\
R &= 6.97 \text{ i} + 1.22 \text{ j}
\end{align*}
\]
Example 10 (Cont.): Find \( \mathbf{i}, \mathbf{j} \) for three vectors:
\[
\mathbf{A} = 5 \text{ m}, 0^0; \quad \mathbf{B} = 2.1 \text{ m}, 20^0; \quad \mathbf{C} = 0.5 \text{ m}, 90^0.
\]

\[
\mathbf{R} = 6.97 \mathbf{i} + 1.22 \mathbf{j}
\]

5. Determine \( R, \theta \) from \( x, y \):

\[
R = \sqrt{(6.97 \text{ m})^2 + (1.22 \text{ m})^2}
\]

\[
R = 7.08 \text{ m}
\]

\[
\tan \phi = \frac{1.22 \text{ m}}{6.97 \text{ m}}
\]

\[
\theta = 9.93^0 \text{ N. of E.}
\]
Example 11: A bike travels 20 m, E then 40 m at 60° N of W, and finally 30 m at 210°. What is the resultant displacement graphically?

Graphically, we use ruler and protractor to draw components, then measure the Resultant $R, \theta$

Let 1 cm = 10 m

$R = (32.6 \text{ m}, 143.0°)$
A Graphical Understanding of the Components and of the Resultant is given below:

Note: $R_x = A_x + B_x + C_x$

$R_y = A_y + B_y + C_y$
Example 11 (Cont.) Using the Component Method to solve for the Resultant.

Write each vector in i,j notation.

\[ A_x = 20 \text{ m}, \quad A_y = 0 \]
\[ A = 20 \text{ i} \]
\[ B_x = -40 \cos 60^\circ = -20 \text{ m} \]
\[ B_y = 40 \sin 60^\circ = +34.6 \text{ m} \]
\[ B = -20 \text{ i} + 34.6 \text{ j} \]
\[ C_x = -30 \cos 30^\circ = -26 \text{ m} \]
\[ C_y = -30 \sin 60^\circ = -15 \text{ m} \]
\[ C = -26 \text{ i} - 15 \text{ j} \]
Example 11 (Cont.) The Component Method

Add algebraically:

\[ A = 20 \, \hat{i} \]
\[ B = -20 \, \hat{i} + 34.6 \, \hat{j} \]
\[ C = -26 \, \hat{i} - 15 \, \hat{j} \]

\[ R = -26 \, \hat{i} + 19.6 \, \hat{j} \]

\[ R = \sqrt{(-26)^2 + (19.6)^2} = 32.6 \, \text{m} \]

\[ \tan \phi = \frac{19.6}{-26} \]
\[ \theta = 143^\circ \]
Example 11 (Cont.) Find the Resultant.

\[ R = -26 \hat{i} + 19.6 \hat{j} \]

The Resultant Displacement of the bike is best given by its polar coordinates \( R \) and \( \theta \).

\[ R = 32.6 \text{ m}; \quad \theta = 143^0 \]
Example 12. Find $A + B + C$ for Vectors Shown below.

$A = 5 \text{ m}, 90^0$
$B = 12 \text{ m}, 0^0$
$C = 20 \text{ m}, -35^0$

$A_x = 0; \quad A_y = +5 \text{ m}$
$B_x = +12 \text{ m}; \quad B_y = 0$
$C_x = (20 \text{ m}) \cos 35^0$
$C_y = -(20 \text{ m}) \sin -35^0$

$A = 0 \text{ i} + 5.00 \text{ j}$
$B = 12 \text{ i} + 0 \text{ j}$
$C = 16.4 \text{ i} - 11.5 \text{ j}$

$R = 28.4 \text{ i} - 6.47 \text{ j}$
Example 12 (Continued).

Find $A + B + C$

\[ R = \sqrt{(28.4 \text{ m})^2 + (6.47 \text{ m})^2} \]

\[ R = 29.1 \text{ m} \]

\[ \tan \phi = \frac{6.47 \text{ m}}{28.4 \text{ m}} \]

\[ \theta = 12.8^0 \text{ S. of E.} \]
Vector Difference

For vectors, signs are indicators of direction. Thus, when a vector is subtracted, the sign (direction) must be changed before adding.

First Consider $A + B$ Graphically:

\[ R = A + B \]
Vector Difference

For vectors, signs are indicators of direction. Thus, when a vector is subtracted, the sign (direction) must be changed before adding.

Now $\mathbf{A} - \mathbf{B}$: First change sign (direction) of $\mathbf{B}$, then add the negative vector.
Addition and Subtraction

Subtraction results in a significant difference both in the magnitude and the direction of the resultant vector. \(|(A - B)| \neq |A| - |B|\)

Comparison of addition and subtraction of B

\[ R = A + B \]

\[ R' = A - B \]
Example 13. Given $A = 2.4 \text{ km}$, $N$ and $B = 7.8 \text{ km}$, $N$: find $A - B$ and $B - A$.

$A - B = (2.43N - 7.74S)$

$B - A = (7.74N - 2.43S)$

$5.31 \text{ km, S}$

$5.31 \text{ km, N}$
Summary for Vectors

- A **scalar quantity** is completely specified by its magnitude only. (40 m, 10 gal)
- A **vector quantity** is completely specified by its magnitude and direction. (40 m, 30°)

**Components of R:**

\[ R_x = R \cos \theta \]
\[ R_y = R \sin \theta \]
Finding the resultant of two perpendicular vectors is like converting from polar \((R, \theta)\) to the rectangular \((R_x, R_y)\) coordinates.

\[
R = \sqrt{x^2 + y^2}
\]

\[
\tan \theta = \frac{y}{x}
\]
Component Method for Vectors

- Start at origin and draw each vector in succession forming a labeled polygon.
- Draw resultant from origin to tip of last vector, noting the quadrant of resultant.
- Write each vector in \( i, j \) notation \((R_x, R_y)\).
- Add vectors algebraically to get resultant in \( i, j \) notation. Then convert to \((R, \theta)\).
Vector Difference

For vectors, signs are indicators of direction. Thus, when a vector is subtracted, the sign (direction) must be changed before adding.

Now $A - B$: First change sign (direction) of $B$, then add the negative vector.
Conclusion of Chapter 3B - Vectors